Spectral Mapping Theorems for Neutron Transport, L^1-Theory

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Abstract

This work deals with spectral mapping theorems for neutron transport semigroups in unbounded geometries and L^1 setting. The mathematical analysis relies on harmonic analysis of certain measure valued mappings related to Dyson-Phillips expansions and on some functional analytic results on the critical spectrum [2, 8].

1 Introduction

The investigation of spectral mapping theorems for neutron transport semigroups

$$\left(e^{t(T+K)}\right)_{t\geq 0}$$

in *unbounded geometries* was initiated recently by the authors in the context of L^p spaces with $1 < p < \infty$ [7], where T and K represent respectively the streaming and the collision operators. The mathematical analysis is based upon two ingredients:
(a) Some functional analytic results on perturbation theory of the critical spectrum of C_0-semigroups [1, 2, 8].
(b) The norm continuity of

$$0 \leq t \mapsto e^{t(T+K)} - e^{tT},$$

i.e. in the *operator norm* topology. The proof of (b) is based on Fourier integral analysis of $e^{t(T+K)} - e^{tT}$ in the case $p = 2$ and on interpolation arguments. The present paper is devoted to the limiting case $p = 1$ which is not covered by [7] and which turns out to be the physical case for neutron transport. Besides the use of the properties of the critical spectrum of perturbed semigroups [2], our mathematical analysis relies on *different tools*. Moreover, we obtain very precise results which are *different* from those given in [7]. Before explaining the content of this paper, it is useful to recall some facts on the critical spectrum of C_0-semigroups [1, 2, 8]. Let X be a Banach space and $\tau = (U(t))_{t\geq 0}$ be a strongly continuous semigroup on X. We consider the Banach space $\widetilde{X} := \ell^\infty(X)$ of all bounded sequences in X endowed with the norm

$$\|\widetilde{x}\| = \sup_{n\in\mathbb{N}} \|x_n\|$$

where $\widetilde{x} = (x_n)_{n\in\mathbb{N}}$. We extend the semigroup $(U(t))_{t\geq 0}$ to $\widetilde{X}$ and obtain a new semigroup $\widetilde{\tau} = \left(\widetilde{U}(t)\right)_{t\geq 0}$ defined by

$$\widetilde{U}(t)\widetilde{x} := (U(t)x_n)_{n\in\mathbb{N}} \quad \text{for } \widetilde{x} = (x_n)_{n\in\mathbb{N}}.$$

Let $\widetilde{X}_\tau$ be the subspace of strong continuity of $\widetilde{\tau}$

$$\widetilde{X}_\tau := \left\{\widetilde{x} \in \widetilde{X};\ \lim_{h\downarrow 0}\left\|\widetilde{U}(h)\widetilde{x} - \widetilde{x}\right\| = 0\right\}.$$

This subspace is closed and $\left(\widetilde{U}(t)\right)_{t\geq 0}$-invariant. On the quotient space $\widehat{X} := \widetilde{X}/\widetilde{X}_\tau$, the semigroup $\left(\widetilde{U}(t)\right)_{t\geq 0}$ induces a quotient semigroup $\widehat{\tau} = \left(\widehat{U}(t)\right)_{t\geq 0}$ given by

$$\widehat{U}(t)\widehat{x} = \widetilde{U}(t)\widetilde{x} + \widetilde{X}_\tau \text{ for } \widehat{x} = \widetilde{x} + \widetilde{X}_\tau.$$

The critical spectrum of $U(t)$ is then defined as

$$\sigma_{crit}(U(t)) = \sigma(\hat{U}(t))$$

while its critical spectral radius is defined as

$$r_{crit}(U(t)) := r(\widehat{U}(t)).$$

Moreover, the critical growth bound is defined as

$$\omega_{crit}(U(\cdot)) := \omega_0(\widehat{U}(\cdot))$$

where ω_0 is the usual growth bound (type). We have:

Theorem 1.1. [**8**] *Let $(U(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X with generator T. Then:*

(a) $\sigma_{crit}(U(t)) \subset \sigma(U(t))$,

(b) $r_{crit}(U(t)) = e^{\omega_{crit}(U(\cdot))t}$,

(c) $\sigma(U(t))\setminus\{0\} = e^{t\sigma(T)} \cup \sigma_{crit}(U(t))\setminus\{0\}$,

(d) $\omega_0(U(\cdot)) = \max\{s(T), \omega_{crit}(U(\cdot))\}$.

Consider now the perturbed semigroup $(V(t))_{t\geq 0}$ generated by $T + K$ where K is a bounded operator:

$$V(t) = \sum_0^\infty U_j(t),$$

where

$$U_0(t) = U(t),\ U_{j+1}(t) = \int_0^t U_0(t-s)KU_j(s)ds\ (j \geq 0). \tag{1.1}$$

The following theorem provides a sufficient condition for the stability of critical growth bound.

Theorem 1.2. [2] *Let $(U(t))_{t\geq 0}$ be a C_0-semigroup with generator T and let $(V(t))_{t\geq 0}$ be the C_0-semigroup generated by $T+K$. If for some $k\in\mathbb{N}$*

$$0<t\mapsto R_k(t):=\sum_{i=k}^{\infty}U_i(t)$$

is norm (right) continuous, then

$$\omega_{crit}(V(\cdot))=\omega_{crit}(U(\cdot)).$$

The stability of critical spectrum is the subject of the next theorem.

Theorem 1.3. [2] *Let $(U(t))_{t\geq 0}$ be a C_0-semigroup with generator T and let $(V(t))_{t\geq 0}$ be the C_0-semigroup generated by $T+K$. If for some $t_0\geq 0$*

$$t_0\leq t\mapsto R_1(t):=V(t)-U(t)$$

is norm (right) continuous, then

$$\sigma_{crit}(V(t))=\sigma_{crit}(U(t))\qquad (t\geq t_0).$$

We give a sufficient condition for recognizing the critical spectrum. We first recall that the approximate spectrum of T is defined by

$$\sigma_{ap}(T):=\{\lambda\in\mathbb{C};\ \ \exists(x_n)_n\subset D(T),\ \|x_n\|=1,\ \|Tx_n-\lambda x_n\|\to 0 \text{ as } n\to\infty\}.$$

Theorem 1.4. [1] *Let $(U(t))_{t\geq 0}$ be a C_0-semigroup with generator T. Let $(\lambda_n)_n\subset\sigma_{ap}(T)$ be such that $\lim\limits_{n\to\infty}|Im\lambda_n|=\infty$ and $\lim\limits_{n\to\infty}e^{t\lambda_n}=\mu$. Then $\mu\in\sigma_{crit}(U(t))$.*

We now present the neutron transport semigroup. Let $\Omega\subset\mathbb{R}^N$ be an open set and let μ be a positive Radon measure on $\mathbb{R}^N$ with support V. We refer to V as the velocity space. The streaming semigroup is given by

$$U(t):L^1(\Omega\times V)\ni\varphi\mapsto e^{-\int_0^t\sigma(x-sv,v)ds}\varphi(x-tv,v)\chi_{\{t<\tau(x,v)\}}\in L^1(\Omega\times V),$$

where $\tau(x,v)=\inf\{s>0;\ \ x-sv\notin\Omega\}$ and $\sigma(\cdot,\cdot)\in L^\infty(\Omega\times V)$ is the collision frequency. Here $\Omega\times V$ is endowed with the product measure $dx\otimes d\mu(v)$. We denote by T the generator of $(U(t))_{t\geq 0}$. The collision operator is the (partial) integral operator

$$K:L^1(\Omega\times V)\ni\varphi\mapsto\int_V k(x,v,v')\varphi(x,v')d\mu(v'),$$

where the scattering kernel $k(\cdot,\cdot,\cdot)$ satisfies the estimate

$$\int_V|k(\cdot,v,\cdot)|d\mu(v)\in L^\infty(\Omega\times V)$$

ensuring the boundedness of K in $L^1(\Omega\times V)$. The neutron transport semigroup is the C_0-semigroup generated by $T+K$. For all the sequel, the collision operator is assumed to be

compact *"with respect to velocities"*, where the compactness is *"collective"* with respect to the space variable. More precisely:
(H1) The family

$$\Big\{ \int_V k(x,v,v')\varphi(v')d\mu(v'); \quad x \in \Omega,\ \varphi \in L^1(V), \|\varphi\|_{L^1(V)} \le 1 \Big\}$$

is relatively compact in $L^1(V)$.
(H2) For each $\psi \in L^\infty(V)$, the family

$$\Big\{ \int_V k(x,v',v)\psi(v')d\mu(v'); \quad x \in \Omega \Big\}$$

is relatively compact in $L^\infty(V)$.
We note that under (H1) and (H2), K can be approximated in the norm operator topology of $\mathcal{L}(L^1(\Omega \times V))$ by collision operators with separable kernels:

$$\sum_{i \in I} \alpha_i(x) f_i(v) g_i(v'), \tag{1.2}$$

where $\alpha_i(\cdot) \in L^\infty(\Omega)$, $f_i(\cdot) \in L^1(V)$, $g_i(\cdot) \in L^\infty(V)$ and I finite (see [7]). We point out that when the scattering kernel is space homogeneous, (H1) and (H2) reduce simply to the compactness of the integral operator

$$L^1(V) \ni \varphi \mapsto \int_V k(v,v')\varphi(v')d\mu(v') \in L^1(V).$$

In this paper, the collision frequency is assumed to be space homogeneous, i.e.

$$\sigma(x,v) = \sigma(v).$$

Our paper is organized as follows: Section 2 is devoted to the neutron transport semigroup in the whole space ($\Omega = \mathbb{R}^N$) with space homogeneous scattering kernels, i.e. $k(x,v,v') = k(v,v')$. We show that if there exists $\alpha > 0$ such that for all $c > 0$ there exists $c' > 0$ such that

$$\sup_{e \in S^{N-1}} \mu \otimes \mu\{(v,v'); |v| \le c, |v'| \le c, |(v-v')\cdot e| \le \varepsilon\} \le c'\varepsilon^\alpha$$

then

$$0 \le t \mapsto R_j(t) \in \mathcal{L}(L^1(\Omega \times V)) \tag{1.3}$$

is norm continuous where j *depends* on α and N. The proof is quite technical and is given in several steps: By a density argument, we can restrict ourselves to the separable case (1.2). In this case, the terms of the Dyson-Phillips expansion (1.1) are shown to be essentially *iterated convolution* of Radon measures depending on time t. In particular, the norm continuity of (1.3) amounts to the fact that such measures depend continuously on t with respect to the *total variation norm.* We show that for $N \ge 2$, regardless of the choice of the velocity measure μ,

$$0 \le t \mapsto R_1(t) = e^{t(T+K)} - e^{tT}$$

is *never* norm continuous. On the other hand, for $N = 1$, we show that

$$0 \leq t \mapsto e^{t(T+K)} - e^{tT}$$

is norm continuous if and only if μ satisfies

$$\sup_{v' \in \mathbb{R}} \mu \left\{ \left[v' - \varepsilon, v' + \varepsilon \right] \right\} \to 0 \text{ as } \varepsilon \to 0.$$

In section 3, we deal with general spatial domains and not (necessarily) space homogeneous scattering kernels under the assumption that the velocity measure μ is "absolutely continuous in speed $|v|$ but arbitrary in directions $\frac{v}{|v|}$" and prove that

$$0 \leq t \mapsto R_2(t)$$

is norm continuous. We note however that such an assumption on μ covers the classical continuous model ($d\mu(v) = dv$) but *not* the multigroup model (Lebesgue measure on spheres). Section 4 is devoted to spectral mapping theorems. We determine first the critical spectrum of the streaming semigroup; we essentially complement some results given in [6, 7]. We derive from the results of Section 2 spectral mapping theorems in the whole space for general velocity measures and space homogeneous scattering kernels. Similarly, we derive from the results of Section 3 a spectral mapping theorem for a restricted class of velocity measures but for general spatial domains and scattering kernels.
The authors thank the referee for helpful remarks and suggestions.

2 On Dyson-Phillips expansions on the whole space

In this section devoted to the case $\Omega = \mathbb{R}^N$, we assume that the scattering kernel is *space homogeneous* and that

(H3) $L^1(V) \ni \varphi \mapsto \int_V k(v, v')\varphi(v')d\mu(v') \in L^1(V)$ is compact.

2.1 Arbitrary dimension

Theorem 2.1. *Let $\Omega = \mathbb{R}^N$. Let μ be a positive (not necessarily finite) Radon measure on $\mathbb{R}^N$ and let (H3) be satisfied. We assume that there exists $\alpha > 0$ such that for all $c > 0$ there exists $c' > 0$ such that*

$$\sup_{e \in S^{N-1}} \mu \otimes \mu\{(v, v'); |v| \leq c, |v'| \leq c, |(v - v') \cdot e| \leq \varepsilon\} \leq c'\varepsilon^{\alpha}. \tag{2.1}$$

Let p_1 be the smallest integer such that $p_1 > \frac{(\alpha+1)}{\alpha}\left(\frac{N}{2} + 1\right)$. Then

$$0 < t \mapsto R_j(t) \in \mathcal{L}(L^1(\mathbb{R}^N \times V))$$

is norm continuous for all $j \geq 2p_1$.

Remark 2.2. Condition (2.1) is obviously satisfied by Lebesgue measures on open sets or on spheres.

The proof of Theorem 2.1 is quite technical and is given in several steps. We observe first that $U_j = [UK]^j * U$ $(j \geq 1)$ where $*$ is the convolution operator which associates to strongly continuous (operator valued) mappings

$$f, g : [0, \infty[\to \mathcal{L}(L^1(\mathbb{R}^N \times V))$$

the strongly continuous mapping

$$f * g : [0, \infty[\ni t \mapsto \int_0^t f(t-s)g(s)ds \in \mathcal{L}(L^1(\mathbb{R}^N \times V))$$

and $[UK]^j := UK * \cdots * UK$ (j times). Here U denotes the mapping $0 \leq t \mapsto U(t)$ and $UK : 0 \leq t \mapsto U(t)K$. We note that: $f, g \mapsto f * g$ is associative. We recall that $0 \leq t \mapsto R_m(t)$ is norm continuous if and only if $0 \leq t \mapsto U_m(t)$ is so [4, Theorem 2.7, p. 18]. Moreover, if $0 \leq t \mapsto [UK]^m(t)$ is norm continuous then $0 < t \mapsto U_m(t)$ is also norm continuous. By the same arguments it suffices to show that $0 \leq t \mapsto K[UK]^{m-1}(t)$ is norm continuous. By density and linearity we may restrict ourselves to

$$K_1U * K_2U * \cdots * K_{m-1}UK_m$$

where K_i $(i = 1, \cdots, m)$ has the form

$$L^1(\mathbb{R}^N \times V) \ni \varphi \mapsto \int_V f_i(v)g_i(v')\varphi(x, v')d\mu(v') \in L^1(\mathbb{R}^N \times V)$$

where $f_i(\cdot) \in L^1(V)$ and $g_i(\cdot) \in L^\infty(V)$. By density again and decomposition we can suppose that f_i and g_i are nonnegative, and f_i are continuous with compact supports. We can then assume without loss of generality that μ has a *compact support*. Let

$$M_i : L^1(\mathbb{R}^N \times \mathbb{R}^N) \ni \varphi \mapsto \int_{\mathbb{R}^N} \varphi(x, v')g_i(v')d\mu(v') = \int_{\mathbb{R}^N} \varphi(x, v')d\mu_i(v') \in L^1(\mathbb{R}^N)$$

where $\mu_i = g_i\mu$. We have

$$K_iM_i = \|\mu_i\| K_i$$

and

$$K_1U * K_2U * \cdots * K_{m-1}UK_m = \prod_{i=1}^{m-1} \|\mu_i\|^{-1} K_1(M_1UK_2 * M_2UK_3 * \cdots * M_{m-1}UK_m).$$

The latter operator is described in:

Lemma 2.3. *Let $m \geq 2$. There exists a finite Radon measure $\beta^m(t)$ on $\mathbb{R}^N$ such that*

$$M_1UK_2 * \cdots * M_mUK_{m+1}\varphi = \beta^m(t) * M_{m+1}\varphi.$$

Proof. We first prove that

$$M_iUK_{i+1}\varphi = \eta_t^i * M_{i+1}\varphi$$

where η_t^i is a finite Radon measure on $\mathbb{R}^N$. Indeed,

$$\begin{aligned}
M_i U K_{i+1}\varphi &= \int_{\mathbb{R}^N} f_{i+1}(v)g_i(v)e^{-t\sigma(v)}M_{i+1}\varphi(x-tv)d\mu(v) \\
&= \int_{\mathbb{R}^N} h_i(v)e^{-t\sigma(v)}M_{i+1}\varphi(x-tv)d\mu(v) \\
&= \int_{\mathbb{R}^N} M_{i+1}\varphi(x-y)d\eta_t^i(y) = \eta_t^i * M_{i+1}\varphi
\end{aligned}$$

where η_t^i is the *image* of $e^{-t\sigma(v)}h_i(v)d\mu$ under the dilation $v \mapsto tv$ and $h_i(v) = f_{i+1}(v)g_i(v)$. Observe that the mapping $0 < t \mapsto \eta_t^i \in \mathcal{M}(\mathbb{R}^N)$ (the space of finite Radon measures) is weak star continuous, i.e., for any $\varphi \in L^1(\mathbb{R}^N)$,

$$0 < t \mapsto \langle \eta_t^i, \varphi \rangle = \int_{\mathbb{R}^N} \varphi(x-tv)e^{-t\sigma(v)}h_i(v)d\mu$$

is continuous. We have

$$\begin{aligned}
M_1UK_2 * M_2UK_3\varphi &= \int_0^t M_1U(t-s)K_2M_2U(s)K_3\varphi ds \\
&= \int_0^t \eta_{t-s}^1 * M_2(M_2U(s)K_3\varphi)ds \\
&= \int_0^t \eta_{t-s}^1 * M_2(\eta_s^2 * M_3\varphi)ds \\
&= \|\mu_2\| \int_0^t \eta_{t-s}^1 * (\eta_s^2 * M_3\varphi)ds \\
&= \|\mu_2\| \int_0^t (\eta_{t-s}^1 * \eta_s^2) * M_3\varphi ds \\
&= \|\mu_2\| \Big[\int_0^t (\eta_{t-s}^1 * \eta_s^2)ds \Big] * M_3\varphi \\
&=: \beta^2(t) * M_3\varphi
\end{aligned}$$

where the integral

$$\beta^2(t) = \|\mu_2\| \int_0^t (\eta_{t-s}^1 * \eta_s^2)ds \tag{2.2}$$

is taken in the weak star sense, i.e.

$$\langle \beta^2(t), \varphi \rangle := \|\mu_2\| \int_0^t \langle \eta_{t-s}^1 * \eta_s^2, \varphi \rangle \, ds.$$

Now, suppose that

$$M_1UK_2 * \cdots * M_{m-1}UK_m\varphi = \beta^{m-1}(t) * M_m\varphi.$$

Then

$$\begin{aligned}
[M_1UK_2 * \cdots * M_mUK_{m+1}](t)\varphi &= \int_0^t [M_1UK_2 * \cdots * M_{m-1}UK_m](t-s)(M_mU(s)K_{m+1}\varphi)ds \\
&= \int_0^t \beta^{m-1}(t-s) * M_m(M_mU(s)K_{m+1}\varphi)ds \\
&= \int_0^t \beta^{m-1}(t-s) * M_m(\eta_s^m * M_{m+1}\varphi)ds \\
&= \|\mu_m\| \int_0^t \beta^{m-1}(t-s) * (\eta_s^m * M_{m+1}\varphi)ds \\
&= \|\mu_m\| \Big[\int_0^t \beta^{m-1}(t-s) * \eta_s^m ds\Big] * M_{m+1}\varphi \\
&=: \beta^m(t) * M_{m+1}\varphi
\end{aligned}$$

where the integral

$$\beta^m(t) = \|\mu_m\| \int_0^t \beta^{m-1}(t-s) * \eta_s^m ds \tag{2.3}$$

is taken in the weak star sense, i.e.

$$\langle \beta^m(t), \varphi\rangle = \|\mu_m\| \int_0^t \left\langle \beta^{m-1}(t-s) * \eta_s^m, \varphi\right\rangle ds.$$

Then $\beta^m(t)$ is defined inductively by (2.3) which ends the proof. ■

Thus, to prove Theorem 2.1, it suffices to prove:

Lemma 2.4. *Let p_1 be the smallest integer such that $p_1 > \frac{\alpha+1}{\alpha}(\frac{N}{2}+1)$. Then for $p \geq p_1$*

$$0 \leq t \mapsto \beta^{2p}(t) \in \mathcal{M}(\mathbb{R}^N)$$

is continuous, where $\mathcal{M}(\mathbb{R}^N)$ (the space of finite Radon measures) is endowed with the total variation norm.

The proof of Lemma 2.4 is given in several steps. Before doing this, as in the proof of Lemma 2.3, we can show, for $p > 1$, that

$$\beta^{2p}(t) = \|\mu_{2p-1}\| \int_0^t \beta^{2(p-1)}(t-s) * \beta^{2p-1,2p}(s)ds \tag{2.4}$$

where

$$\beta^{i,i+1}(t) = \|\mu_{i+1}\| \int_0^t \eta_{t-s}^i * \eta_s^{i+1} ds.$$

Let us show first that for p large enough $\beta^{2p}(t)$ is a *function*, i.e. $\beta^{2p}(t)$ is absolutely continuous with respect to Lebesgue measure.

Lemma 2.5. *Let p_0 be the smallest integer such that $p_0 > \frac{N(\alpha+1)}{2\alpha}$. Then for all $p \geq p_0$ we have $\beta^{2p}(t) \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ for all $t \in [0,T]$.*

Proof. We start with $\beta^2(t)$ (see (2.2)). We have

$$\begin{aligned}
&\|\mu_2\|^{-1}\,\widehat{\beta^2(t)}(\xi) \\
= \;& (2\pi)^{N/2} \int_0^t \widehat{\eta_{t-s}^1}(\xi)\widehat{\eta_s^2}(\xi) ds \\
= \;& (2\pi)^{-N/2} \int_0^t \Big[\int_{\mathbb{R}^N} e^{-iv\cdot\xi} d\eta_{t-s}^1(v)\Big]\Big[\int_{\mathbb{R}^N} e^{-iv'\cdot\xi} d\eta_s^2(v')\Big] ds \\
= \;& (2\pi)^{-N/2} \int_0^t \Big[\int_{\mathbb{R}^N} e^{-i(t-s)v\cdot\xi} e^{-(t-s)\sigma(v)} h_1(v) d\mu(v)\Big]\Big[\int_{\mathbb{R}^N} e^{-isv'\cdot\xi} e^{-s\sigma(v')} h_2(v') d\mu(v')\Big] ds \\
= \;& (2\pi)^{-N/2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big[\int_0^t e^{-i(t-s)v\cdot\xi} e^{-(t-s)\sigma(v)} e^{-isv'\cdot\xi} e^{-s\sigma(v')} ds\Big] h_1(v) h_2(v') d\mu(v) d\mu(v').
\end{aligned}$$

Introducing polar coordinates $\xi = |\xi|\, e$, $e \in S^{N-1}$, we decompose the last integral as

$$\begin{aligned}
&\iint_{|(v'-v)\cdot e|\leq\varepsilon} \Big[\int_0^t e^{-i(t-s)v\cdot\xi} e^{-(t-s)\sigma(v)} e^{-isv'\cdot\xi} e^{-s\sigma(v')} ds\Big] h_1(v) h_2(v') d\mu(v) d\mu(v') \\
+ &\iint_{|(v'-v)\cdot e|>\varepsilon} \Big[\int_0^t e^{-i(t-s)v\cdot\xi} e^{-(t-s)\sigma(v)} e^{-isv'\cdot\xi} e^{-s\sigma(v')} ds\Big] h_1(v) h_2(v') d\mu(v) d\mu(v')
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary. We have

$$\begin{aligned}
&\Big| \iint_{|(v'-v)\cdot e|\leq\varepsilon} \Big[\int_0^t e^{-i(t-s)v\cdot\xi} e^{-(t-s)\sigma(v)} e^{-isv'\cdot\xi} e^{-s\sigma(v')} ds\Big] h_1(v) h_2(v') d\mu(v) d\mu(v') \Big| \\
\leq \;& t e^{2t\|\sigma\|_\infty} \|h_1 h_2\|_\infty \iint_{|(v'-v)\cdot e|\leq\varepsilon} d\mu(v) d\mu(v').
\end{aligned}$$

We estimate the second integral

$$\Big| \iint_{|(v'-v)\cdot e|>\varepsilon} \Big[\int_0^t e^{-i(t-s)v\cdot\xi} e^{-(t-s)\sigma(v)} e^{-isv'\cdot\xi} e^{-s\sigma(v')} ds\Big] h_1(v) h_2(v') d\mu(v) d\mu(v') \Big|$$

$$\leq \iint\limits_{|(v'-v)\cdot e|>\varepsilon} \frac{\left|e^{-itv'\cdot\xi}e^{-t\sigma(v')} - e^{-itv\cdot\xi}e^{-t\sigma(v)}\right|}{|i(v-v')\cdot\xi + \sigma(v) - \sigma(v')|} \left|h_1(v)h_2(v')\right| d\mu(v)d\mu(v')$$
$$\leq \iint\limits_{|(v'-v)\cdot e|>\varepsilon} \frac{2\left|h_1(v)h_2(v')\right|}{|(v-v')\cdot e|\,|\xi|} d\mu(v)d\mu(v')$$
$$\leq \frac{2}{\varepsilon\,|\xi|} \iint\limits_{|(v'-v)\cdot e|>\varepsilon} \left|h_1(v)h_2(v')\right| d\mu(v)d\mu(v') \leq \frac{2\|h_1\|_1\|h_2\|_1}{\varepsilon\,|\xi|}.$$

Thus

$$\left|\widehat{\beta^2(t)}(\xi)\right| \leq (2\pi)^{-N/2}\left\|\mu_2\right\| \Big(Te^{2T\|\sigma\|_\infty}\left\|h_1h_2\right\|_\infty \iint\limits_{|(v'-v)\cdot e|\leq\varepsilon} d\mu(v)d\mu(v') + \frac{2\left\|h_1\right\|_1\|h_2\|_1}{\varepsilon\,|\xi|}\Big).$$

Let $0<\tau<1$ and $\varepsilon = |\xi|^{-\tau}$ then

$$\begin{aligned}\left|\widehat{\beta^2(t)}(\xi)\right| &\leq (2\pi)^{-N/2}\left\|\mu_2\right\|(Te^{2T\|\sigma\|_\infty}\left\|h_1h_2\right\|_\infty + 2\left\|h_1\right\|_1\|h_2\|_1)(a(\xi)+b(\xi)) \\ &=: C_1(a(\xi)+b(\xi)),\end{aligned}$$

where

$$a(\xi) := \sup_{e\in S^{N-1}} \mu\otimes\mu\left\{(v,v');\quad \left|(v^{'}-v)\cdot e\right| \leq |\xi|^{-\tau}\right\} \text{ and } b(\xi) := \frac{1}{|\xi|^{1-\tau}}.$$

Consider

$$\beta^{2p-1,2p}(t) = \|\mu_{2p}\|\int_0^t \eta_{t-s}^{2p-1} * \eta_s^{2p} ds;$$

as previously we can prove that

$$\left|\widehat{\beta^{2p-1,2p}}(t)(\xi)\right| \leq (2\pi)^{-N/2}\|\mu_{2p}\|(Te^{2T\|\sigma\|_\infty}\left\|h_{2p-1}h_{2p}\right\|_\infty + 2\left\|h_{2p-1}\right\|_1\|h_{2p}\|_1)(a(\xi)+b(\xi)) \quad (2.5)$$

Now let us prove by induction that there exists $C_p>0$ (depending only on p) such that

$$\left|\widehat{\beta^{2p}(t)}(\xi)\right| \leq C_p(a(\xi)+b(\xi))^p \text{ for all } t\in[0,T].$$

Suppose that there exists $C_{p-1}>0$ such that

$$\left|\widehat{\beta^{2(p-1)}}(t)(\xi)\right| \leq C_{p-1}(a(\xi)+b(\xi))^{p-1} \text{ for all } t\in[0,T].$$

Then by (2.4) and (2.5) we have

$$\begin{aligned}&\|\mu_{2p-1}\|^{-1}\left|\widehat{\beta^{2p}(t)}(\xi)\right| \\ \leq\ & (2\pi)^{N/2}\int_0^t \left|\widehat{\beta^{2(p-1)}}(t-s)(\xi)\widehat{\beta^{2p-1,2p}}(s)(\xi)\right|ds \\ \leq\ & tC_{p-1}(a(\xi)+b(\xi))^{p-1}\|\mu_{2p}\|(Te^{2T\|\sigma\|_\infty}\left\|h_{2p-1}h_{2p}\right\|_\infty + 2\|h_{2p-1}\|_1\|h_{2p}\|_1)(a(\xi)+b(\xi)).\end{aligned}$$

Then

$$\left|\widehat{\beta^{2p}(t)}(\xi)\right| \le C_p(a(\xi)+b(\xi))^p \text{ for all } t\in[0,T] \tag{2.6}$$

where $C_p := \|\mu_{2p-1}\|\|\mu_{2p}\|TC_{p-1}\Big(Te^{2T\|\sigma\|_\infty}\|h_{2p-1}h_{2p}\|_\infty + 2\|h_{2p-1}\|_1\|h_{2p}\|_1\Big)$. Put $\tau=\frac{1}{1+\alpha}$ (clearly $0<\tau<1$) using (2.1) we obtain

$$\left|\widehat{\beta^{2p}(t)}(\xi)\right| \le C_p[\frac{c'}{|\xi|^{\alpha\tau}}+\frac{1}{|\xi|^{1-\tau}}]^p \le \frac{2^pC_p(\max\{c',1\})^p}{|\xi|^{\frac{p\alpha}{1+\alpha}}}.$$

Hence $\widehat{\beta^{2p}(t)}\in L^2(\mathbb{R}^N)$ for all $p\ge p_0>\frac{N(1+\alpha)}{2\alpha}$ and for all $t\in[0,T]$. By Parseval's identity we have $\beta^{2p}(t)\in L^2(\mathbb{R}^N)$ for all $p\ge p_0>\frac{N(1+\alpha)}{2\alpha}$ and for all $t\in[0,T]$. Since $\beta^{2p}(t)$ is also a *bounded* Radon measure on $\mathbb{R}^N$ we conclude that $\beta^{2p}(t)\in L^1(\mathbb{R}^N)$. ■

Now, the proof of Lemma 2.4 amounts to

$$0\le t\mapsto \beta^{2p}(t)\in L^1(\mathbb{R}^N) \text{ is continuous.} \tag{2.7}$$

We deal first with the continuity in L^2 norm.

Lemma 2.6. *Let p_1 be the smallest integer such that $p_1>\frac{\alpha+1}{\alpha}(\frac{N}{2}+1)$. Then for all $p\ge p_1$, $]0,T]\ni t\mapsto\beta^{2p}(t)\in L^2(\mathbb{R}^N)$ is continuous.*

Proof. We note the following elementary estimates which will be used repeatedly in the sequel. There exists $C>1$ such that

$$\left|e^{-itv\cdot\xi-t\sigma(v)}-e^{-i\bar{t}v\cdot\xi-\bar{t}\sigma(v)}\right|\le C\max\{1,|\xi|\}\left|t-\bar{t}\right| \tag{2.8}$$

and

$$\left|e^{-it(v-v')\cdot\xi-t(\sigma(v)-\sigma(v'))}-e^{-i\bar{t}(v-v')\cdot\xi-\bar{t}(\sigma(v)-\sigma(v'))}\right|\le C\max\{1,|\xi|\}\left|t-\bar{t}\right| \tag{2.9}$$

for all $t,\ \bar{t}\in[0,T]$, for almost all $v,\ v'\in V$ and for all $\xi\in\mathbb{R}^N$.

In a first step we prove inductively that

$$\left|\widehat{\beta^{2p}(t)}(\xi)-\widehat{\beta^{2p}(\bar{t})}(\xi)\right|\le C_p^*\left|t-\bar{t}\right|\,|\xi|\,(a(\xi)+b(\xi))^p$$

for all ξ such that $|\xi|\ge1$, where C_p^* is some constant which depends only on p. We recall that

$$a(\xi):=\sup_{e\in S^{N-1}}\mu\otimes\mu\left\{(v,v');\ \ \left|(v'-v)\cdot e\right|\le|\xi|^{-\tau}\right\} \text{ and } b(\xi):=\frac{1}{|\xi|^{1-\tau}}.$$

Using the expression of $\widehat{\beta^2(t)}(\xi)$ given in the proof of Lemma 2.5 we have

$$\begin{aligned}
&\|\mu_2\|^{-1}\left(\widehat{\beta^2(t)}(\xi)-\widehat{\beta^2(\bar{t})}(\xi)\right)\\
=\ &(2\pi)^{-N/2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}h_1(v)h_2(v')d\mu(v)d\mu(v')\\
&\times\Big[\int\limits_0^t e^{-i(t-s)v\cdot\xi-(t-s)\sigma(v)}e^{-isv'\cdot\xi-s\sigma(v')}ds-\int\limits_0^{\bar{t}} e^{-i(\bar{t}-s)v\cdot\xi-(\bar{t}-s)\sigma(v)}e^{-isv'\cdot\xi-s\sigma(v')}\Big]ds
\end{aligned}$$

$$
\begin{aligned}
= \quad & (2\pi)^{-N/2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h_1(v)h_2(v')d\mu(v)d\mu(v') \\
& \times\Big[e^{-itv\cdot\xi-t\sigma(v)} \int_{\bar t}^{t} e^{-is(v'-v)\cdot\xi-s(\sigma(v')-\sigma(v))}ds \\
& \qquad +(e^{-itv\cdot\xi-t\sigma(v)} - e^{-i\bar{t}v\cdot\xi-\bar{t}\sigma(v)}) \int_0^{\bar t} e^{-is(v'-v)\cdot\xi-s(\sigma(v')-\sigma(v))}ds\Big]. \qquad (2.10)
\end{aligned}
$$

Introducing polar coordinates $\xi = |\xi|\, e$, $e \in S^{N-1}$, we decompose the last integral as

$$
\begin{aligned}
& \iint_{|(v'-v)\cdot e|\le|\xi|^{-\tau}} h_1(v)h_2(v')d\mu(v)d\mu(v') \\
& \qquad \times\Big[e^{-itv\cdot\xi-t\sigma(v)} \int_{\bar t}^{t} e^{-is(v-v)\cdot\xi-s(\sigma(v')-\sigma(v))}ds \\
& \qquad\qquad +(e^{-itv\cdot\xi-t\sigma(v)} - e^{-i\bar{t}v\cdot\xi-\bar{t}\sigma(v)}) \int_0^{\bar t} e^{-is(v'-v)\cdot\xi-s(\sigma(v')-\sigma(v))}ds\Big] \\
+ \quad & \iint_{|(v'-v)\cdot e|>|\xi|^{-\tau}} h_1(v)h_2(v')d\mu(v)d\mu(v') \\
& \qquad \times\Big[e^{-itv\cdot\xi-t\sigma(v)} \int_{\bar t}^{t} e^{-is(v-v)\cdot\xi-s(\sigma(v')-\sigma(v))}ds \\
& \qquad\qquad +(e^{-itv\cdot\xi-t\sigma(v)} - e^{-i\bar{t}v\cdot\xi-\bar{t}\sigma(v)}) \int_0^{\bar t} e^{-is(v'-v)\cdot\xi-s(\sigma(v')-\sigma(v))}ds\Big] \\
=: \quad & I_1 + I_2.
\end{aligned}
$$

Clearly

$$
\begin{aligned}
& \Big| \iint_{|(v'-v)\cdot e|\le|\xi|^{-\tau}} h_1(v)h_2(v')d\mu(v)d\mu(v')e^{-itv\cdot\xi-t\sigma(v)} \int_{\bar t}^{t} e^{-is(v'-v)\cdot\xi-s(\sigma(v')-\sigma(v))}ds\Big| \\
\le \quad & e^{2T\|\sigma\|_\infty}|t-\bar t| \iint_{|(v'-v)\cdot e|\le|\xi|^{-\tau}} \big|h_1(v)h_2(v')\big|\, d\mu(v)d\mu(v') \\
\le \quad & e^{2T\|\sigma\|_\infty}\left\|h_1h_2\right\|_\infty \big|t-\bar t\big|a(\xi). \qquad (2.11)
\end{aligned}
$$

Using (2.8) we obtain

$$
\Big| \iint_{|(v'-v).e|\le|\xi|^{-\tau}} h_1(v)h_2(v')d\mu(v)d\mu(v')(e^{-itv\cdot\xi-t\sigma(v)} - e^{-i\bar{t}v\cdot\xi-\bar{t}\sigma(v)})
$$

$$\times \int_0^{\bar t} e^{-is(v'-v)\cdot\xi - s(\sigma(v')-\sigma(v))} ds \Big|$$

$$\le C|\xi|\left|t-\bar t\right| T e^{2T\|\sigma\|_\infty} \iint_{|(v'-v)\cdot e|\le|\xi|^{-\tau}} \left|h_1(v)h_2(v')\right| d\mu(v)d\mu(v')$$

$$\le CTe^{2T\|\sigma\|_\infty}\|h_1h_2\|_\infty a(\xi)|\xi|\left|t-\bar t\right|. \tag{2.12}$$

Then adding (2.11) and (2.12) we get

$$|I_1| \le (1+CT)e^{2T\|\sigma\|_\infty}\|h_1h_2\|_\infty a(\xi) \ |\xi|\left|t-\bar t\right| \quad \forall|\xi|\ge 1. \tag{2.13}$$

Consider I_2. First we have by (2.9)

$$\Big| \iint_{|(v'-v)\cdot e|>|\xi|^{-\tau}} h_1(v)h_2(v')d\mu(v)d\mu(v') e^{-itv\cdot\xi - t\sigma(v)} \int_{\bar t}^{t} e^{-is(v'-v)\cdot\xi - s(\sigma(v')-\sigma(v))} ds\Big|$$

$$\le \iint_{|(v'-v)\cdot e|>|\xi|^{-\tau}} \left|h_1(v)h_2(v')\right| \left|\frac{e^{-it(v'-v)\cdot\xi - t(\sigma(v')-\sigma(v))} - e^{-i\bar t(v'-v)\cdot\xi - \bar t(\sigma(v')-\sigma(v))}}{i(v-v')\cdot\xi + \sigma(v)-\sigma(v')}\right| d\mu(v)d\mu(v^{'})$$

$$\le \iint_{|(v'-v)\cdot e|>|\xi|^{-\tau}} \left|h_1(v)h_2(v')\right| \frac{C|\xi|\left|t-\bar t\right|}{|(v-v')\cdot e|\,|\xi|} d\mu(v)d\mu(v')$$

$$\le \iint_{|(v'-v)\cdot e|>|\xi|^{-\tau}} \left|h_1(v)h_2(v')\right| \frac{C|\xi|\left|t-\bar t\right|}{|\xi|^{-\tau}\,|\xi|} d\mu(v)d\mu(v')$$

$$\le C\|h_1\|_1\|h_2\|_1 b(\xi)|\xi|\left|t-\bar t\right|. \tag{2.14}$$

Similarly, applying (2.8) we obtain

$$\Big| \iint_{|(v'-v)\cdot e|>|\xi|^{-\tau}} h_1(v)h_2(v')d\mu(v)d\mu(v')(e^{-itv\cdot\xi - t\sigma(v)} - e^{-i\bar t v\cdot\xi - \bar t\sigma(v)})$$

$$\times \int_0^{\bar t} e^{-is(v'-v)\cdot\xi - s(\sigma(v')-\sigma(v))} ds\Big|$$

$$\le \iint_{|(v'-v)\cdot e|>|\xi|^{-\tau}} \left|h_1(v)h_2(v')\right| \left|e^{-itv\cdot\xi - t\sigma(v)} - e^{-i\bar t v\cdot\xi - \bar t\sigma(v)}\right|$$

$$\times\left|\frac{e^{-i\bar t(v'-v)\cdot\xi - \bar t(\sigma(v')-\sigma(v))} - 1}{i(v-v')\cdot\xi + \sigma(v)-\sigma(v')}\right| d\mu(v)d\mu(v')$$

$$\le (e^{2T\|\sigma\|_\infty}+1)C\|h_1\|_1\|h_2\|_1 b(\xi)|\xi|\left|t-\bar t\right|. \tag{2.15}$$

Thus adding (2.14) and (2.15)

$$|I_2| \le (e^{2T\|\sigma\|_\infty}+2)C\|h_1\|_1\|h_2\|_1 b(\xi) \ |\xi|\left|t-\bar t\right| \quad \forall|\xi|\ge 1. \tag{2.16}$$

By (2.13) and (2.16) we have

$$\left|\widehat{\beta^2(t)}(\xi) - \widehat{\beta^2(\bar t)}(\xi)\right| \le C_1^*(a(\xi)+b(\xi))|\xi|\left|t-\bar t\right| \quad \forall|\xi|\ge 1$$

where $C_1^* := (2\pi)^{-N/2}\|\mu_2\|\Big[(1+CT)e^{2T\|\sigma\|_\infty}\left\|h_1h_2\right\|_\infty + (e^{2T\|\sigma\|_\infty}+2)C\|h_1\|_1\|h_2\|_1\Big]$. Let us show that there exists C_p^* which depends only on p such that

$$\left|\widehat{\beta^{2p}(t)}(\xi) - \widehat{\beta^{2p}(\overline{t})}(\xi)\right| \le C_p^* \left|\xi\right| \left|t-\overline{t}\right|(a(\xi)+b(\xi))^p \text{ for } |\xi| \ge 1 \text{ and } t, \overline{t} \in [0,T].$$

Suppose that

$$\left|\beta^{2\widehat{(p-1)}}(t)(\xi) - \beta^{2\widehat{(p-1)}}(\overline{t})(\xi)\right| \le C_{p-1}^* \left|\xi\right| \left|t-\overline{t}\right| (a(\xi)+b(\xi))^{p-1}$$

for $|\xi| \ge 1$ and $t, \overline{t} \in [0,T]$. Thanks to (2.4), (2.5) and (2.6) we have

$$\begin{aligned}
&\|\mu_{2p-1}\|^{-1}\left|\widehat{\beta^{2p}(t)}(\xi) - \widehat{\beta^{2p}(\overline{t})}(\xi)\right| \\
\le\ & (2\pi)^{N/2}\int_0^{\overline{t}} \left|\beta^{2\widehat{(p-1)}(t}-s)(\xi) - \beta^{2\widehat{(p-1)}(\overline{t}}-s)(\xi)\right|\left|\beta^{2\widehat{p-1,2p}}(s)(\xi)\right|ds \\
&+(2\pi)^{N/2}\int_{\overline{t}}^{t} \left|\beta^{2\widehat{(p-1)}(t}-s)(\xi)\right|\left|\beta^{2\widehat{p-1,2p}}(s)(\xi)\right|ds \\
\le\ & \overline{t}C_{p-1}^*\|\mu_{2p}\|\left(Te^{2T\|\sigma\|_\infty}\left\|h_{2p-1}h_{2p}\right\|_\infty + 2\|h_{2p-1}\|_1\|h_{2p}\|_1\right)|\xi|\left|t-\overline{t}\right|(a(\xi)+b(\xi))^p \\
&+C_{p-1}\|\mu_{2p}\|\left(Te^{2T\|\sigma\|_\infty}\left\|h_{2p-1}h_{2p}\right\|_\infty + 2\|h_{2p-1}\|_1\|h_{2p}\|_1\right)\left|t-\overline{t}\right|(a(\xi)+b(\xi))^p.
\end{aligned}$$

Then

$$\left|\widehat{\beta^{2p}(t)}(\xi) - \widehat{\beta^{2p}(\overline{t})}(\xi)\right| \le C_p^* \left|\xi\right| \left|t-\overline{t}\right|(a(\xi)+b(\xi))^p \tag{2.17}$$

for $|\xi| \ge 1$ and $t, \overline{t} \in [0,T]$ where

$$C_p^* := \|\mu_{2p-1}\|\|\mu_{2p}\|\left(Te^{2T\|\sigma\|_\infty}\left\|h_{2p-1}h_{2p}\right\|_\infty + 2\|h_{2p-1}\|_1\|h_{2p}\|_1\right)\left(TC_{p-1}^* + C_{p-1}\right).$$

We are going to estimate

$$\left|\widehat{\beta^{2p}(t)}(\xi) - \widehat{\beta^{2p}(\overline{t})}(\xi)\right| \text{ for } |\xi| < 1.$$

Using (2.8) and decomposition (2.10) we obtain

$$\begin{aligned}
\left|\widehat{\beta^2(t)}(\xi) - \widehat{\beta^2(\overline{t})}(\xi)\right| &\le (2\pi)^{-N/2}\|\mu_2\|\left(e^{2T\|\sigma\|_\infty}\|h_1\|_1\|h_2\|_1 + TCe^{2T\|\sigma\|_\infty}\|h_1\|_1\|h_2\|_1\right)\left|t-\overline{t}\right| \\
&=: C_{*1}\left|t-\overline{t}\right| \text{ for } |\xi| < 1.
\end{aligned}$$

Inductively we can obtain

$$\left|\widehat{\beta^{2p}(t)}(\xi) - \widehat{\beta^{2p}(\overline{t})}(\xi)\right| \le C_{*p}\left|t-\overline{t}\right| \text{ for } |\xi| < 1 \tag{2.18}$$

where $C_{*p} > 0$ depends only on p. Considering (2.17), choosing $\tau = \frac{1}{1+\alpha}$ and using (2.1) and (2.18) we obtain

$$\int_{\mathbb{R}^N} \left|\widehat{\beta^{2p}(t)}(\xi) - \widehat{\beta^{2p}(\overline{t})}(\xi)\right|^2 d\xi$$

$$
\begin{aligned}
&\leq \int_{|\xi|<1} \left|\widehat{\beta^{2p}(t)}(\xi) - \widehat{\beta^{2p}(\bar{t})}(\xi)\right|^2 d\xi + \int_{|\xi|\geq 1} \left|\widehat{\beta^{2p}(t)}(\xi) - \widehat{\beta^{2p}(\bar{t})}(\xi)\right|^2 d\xi \\
&\leq C_{*p}^2 \left|t-\bar{t}\right|^2 vol(B(0,1)) + \left(2^p C_p^* (\max\{c',1\})^p\right)^2 \left|t-\bar{t}\right|^2 \int_{|\xi|\geq 1} |\xi|^{2(1-p\frac{\alpha}{1+\alpha})} d\xi \\
&\leq \left|t-\bar{t}\right|^2 \Big[C_{*p}^2 vol(B(0,1)) + \left(2^p C_p^* (\max\{c',1\})^p\right)^2 \int_{|\xi|\geq 1} |\xi|^{2(1-p\frac{\alpha}{1+\alpha})} d\xi\Big]
\end{aligned}
$$

for $p \geq p_1 > \frac{1+\alpha}{\alpha}(\frac{N}{2}+1)$ where $vol(B(0,1))$ is the volume of unit ball of $\mathbb{R}^N$. ■

Before proving (2.7) we need the following two lemmas.

Lemma 2.7. *Let ν be a positive Radon measure on $\mathbb{R}^N$ with compact support and $(\beta(t))_{t\geq 0}$ be a family of positive measures uniformly bounded in $t \in [0,T]$ (for all $T>0$) and such that*

$$
\beta(t)\{\mathbb{R}^N \backslash B(0,n)\} \to 0 \text{ as } n \to \infty \tag{2.19}
$$

uniformly in $t \in [\delta, \frac{1}{\delta}]$ for all $\delta > 0$ where $B(0,n)$ is the ball centred at 0 with radius n. Then

$$
\int_0^t \beta(t-s) * \nu_s ds\{\mathbb{R}^N \backslash B(0,n)\} \to 0 \text{ as } n \to \infty
$$

uniformly in $t \in [0,T]$ for all $T>0$ where ν_s is the image of ν under the dilation $v \mapsto sv$.

Proof. Let $T>0$, $\varepsilon>0$ and $t \in [0,T]$. Set $M = \sup_{t\in[0,T]} \|\beta(t)\|$. By definition we have

$$
\begin{aligned}
\beta(t-s) * \nu_s\{\mathbb{R}^N \backslash B(0,n)\} &= \iint_{\mathbb{R}^N\times\mathbb{R}^N} \chi_{\mathbb{R}^N\backslash B(0,n)}(x+y) d\nu_s(x) d\beta(t-s)(y) \\
&= \int_{\mathbb{R}^N} \beta(t-s)\{-x + \mathbb{R}^N \backslash B(0,n)\} d\nu_s(x) \\
&= \int_{\mathbb{R}^N} \beta(t-s)\{-sx + \mathbb{R}^N \backslash B(0,n)\} d\nu(x).
\end{aligned} \tag{2.20}
$$

We note that

$$
\begin{aligned}
&\int_0^t \beta(t-s) * \nu_s ds\{\mathbb{R}^N \backslash B(0,n)\} \\
= &\int_0^{\varepsilon} \beta(t-s) * \nu_s ds\{\mathbb{R}^N \backslash B(0,n)\} + \int_{\varepsilon}^{t-\varepsilon} \beta(t-s) * \nu_s ds\{\mathbb{R}^N \backslash B(0,n)\} \\
&+ \int_{t-\varepsilon}^{t} \beta(t-s) * \nu_s ds\{\mathbb{R}^N \backslash B(0,n)\} \\
=: &\; I_1 + I_2 + I_3.
\end{aligned}
$$

From (2.20)

$$I_1 \leq \varepsilon M \|\nu\| \quad \text{and} \quad I_3 \leq \varepsilon M \|\nu\|. \tag{2.21}$$

On the other hand, there exists a compact set C_ε such that

$$\text{Supp } \nu_s \subset C_\varepsilon \quad \text{for all} \quad s \in [\varepsilon, T],$$

where Supp ν denotes the support of ν. It follows that there exists an integer n_0 such that

$$-x + \mathbb{R}^N \backslash B(0, n) \subset \mathbb{R}^N \backslash B(0, \frac{n}{2})$$

for all $n \geq n_0$ and for all $x \in C_\varepsilon$ and then

$$\begin{aligned} I_2 &\leq \int_\varepsilon^{t-\varepsilon} \int_{\mathbb{R}^N} \beta(t-s)\{\mathbb{R}^N \backslash B(0, \frac{n}{2})\} d\nu_s(x) ds \\ &\leq \sup_{r \in [\varepsilon, T]} \beta(r)\{\mathbb{R}^N \backslash B(0, \frac{n}{2})\} \int_\varepsilon^{t-\varepsilon} \int_{\mathbb{R}^N} d\nu_s(x) ds \\ &\leq T\|\nu\| \sup_{r \in [\varepsilon, T]} \beta(r)\{\mathbb{R}^N \backslash B(0, \frac{n}{2})\}. \end{aligned}$$

Thanks to (2.19) there exists an integer $n_1 \geq n_0$ such that

$$I_2 \leq T\|\nu\|\varepsilon \tag{2.22}$$

for all $n \geq n_1$ and for all $t \in [\varepsilon, T]$. Using (2.21) and (2.22) we obtain

$$I_1 + I_2 + I_3 \leq \varepsilon \|\nu\|(2M + T)$$

for all $n \geq n_1$ and $t \in [0, T]$. ■

Lemma 2.8. *For all $m \geq 2$ we have*

$$\beta^m(t)\{\mathbb{R}^N \backslash B(0, n)\} \to 0 \text{ as } n \to \infty \tag{2.23}$$

uniformly in $t \in [0, T]$ for all $T > 0$.

Proof. Taking advantage of the expression (2.3), i.e.

$$\beta^m(t) = \|\mu_m\| \int_0^t \beta^{m-1}(t-s) * \eta_s^m ds$$

we apply inductively Lemma 2.7 to show (2.23). Indeed, we first observe that for $T > 0$

$$\sup_{t \in [0,T]} \|\beta^m(t)\| < \infty \tag{2.24}$$

which is true for $m = 2$ since (2.2) shows

$$\|\beta^2(t)\| \leq T\|\mu_2\|\|h_1\|_1\|h_2\|_1$$

for $t \in [0,T]$. The proof of (2.24) follows by induction. Let us show that η^1_s satisfies (2.19). Since η^1 has compact support, there exists a compact set C_δ, which depends only on δ, such that Supp $\eta^1_s \subset C_\delta$ for all $s \in [\delta, \frac{1}{\delta}]$. Thus Supp $\eta^1_s \subset C_\delta \subset B(0,n)$ for n *large enough* and then

$$\eta^1_s\{\mathbb{R}^N \backslash B(0,n)\} = 0 \;\text{ for all } s \in [\delta, \frac{1}{\delta}].$$

Combining this with the uniform boundedness of $(\eta^1_s)_{s\geq 0}$ and Lemma 2.7 we get

$$\beta^2(t)\{\mathbb{R}^N \backslash B(0,n)\} \to 0 \text{ as } n \to \infty \tag{2.25}$$

uniformly on compact intervals of $[0,+\infty[$. Now, using (2.3), (2.24) and (2.25), it is easy to end the proof by induction. ∎

We are now ready to prove Lemma 2.4.

Proof of Lemma 2.4 It suffices to show (2.7). Let $\bar{t} > 0$ and $\varepsilon > 0$. From Lemma 2.8 there exists n_0 such that

$$\beta^{2p_1}(t)\{\mathbb{R}^N \backslash B(0,n_0)\} \leq \frac{\varepsilon}{3} \text{ uniformly in } t \in \left[\bar{t}-\delta, \bar{t}+\delta\right]$$

(with a suitable choice of δ). Then

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left|\beta^{2p_1}(t)(x) - \beta^{2p_1}(\bar{t})(x)\right| dx \\
\leq\; & \int_{B(0,n_0)} \left|\beta^{2p_1}(t)(x) - \beta^{2p_1}(\bar{t})(x)\right| dx + \int_{\mathbb{R}^N \backslash B(0,n_0)} \left|\beta^{2p_1}(t)(x) - \beta^{2p_1}(\bar{t})(x)\right| dx \\
\leq\; & [vol(B(0,n_0))]^{\frac{1}{2}} \left[\int_{B(0,n_0)} \left|\beta^{2p_1}(t)(x) - \beta^{2p_1}(\bar{t})(x)\right|^2 dx\right]^{\frac{1}{2}} \\
&+ \int_{\mathbb{R}^N \backslash B(0,n_0)} \beta^{2p_1}(t)(x)dx + \int_{\mathbb{R}^N \backslash B(0,n_0)} \beta^{2p_1}(\bar{t})(x)dx \\
\leq\; & [vol(B(0,n_0))]^{\frac{1}{2}} \left[\int_{\mathbb{R}^N} \left|\beta^{2p_1}(t)(x) - \beta^{2p_1}(\bar{t})(x)\right|^2 dx\right]^{\frac{1}{2}} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}
\end{aligned}$$

for $t \in \left[\bar{t}-\delta, \bar{t}+\delta\right]$. By Lemma 2.6 there exists $\delta_1 > 0$ such that

$$\left[\int_{\mathbb{R}^N} \left|\beta^{2p_1}(t)(x) - \beta^{2p_1}(\bar{t})(x)\right|^2 dx\right]^{\frac{1}{2}} \leq \frac{\varepsilon}{3vol(B(0,n_0))}$$

for $t \in \left[\bar{t}-\delta_1, \bar{t}+\delta_1\right]$. Thus

$$\int_{\mathbb{R}^N} \left|\beta^{2p_1}(t)(x) - \beta^{2p_1}(\bar{t})(x)\right| dx \leq \varepsilon$$

for $t \in \left[\bar{t}-\delta_1, \bar{t}+\delta_1\right]$ which ends the proof. ∎

Remark 2.9. It is an open problem to prove Theorem 2.1 when $\Omega \neq \mathbb{R}^N$ or when the scattering kernels are not space homogeneous.

In the L^p theory $(1 < p < \infty)$, $0 \leq t \mapsto R_1(t)$ is norm continuous [7]. We now show that this is *never* true in L^1.

Theorem 2.10. *Let μ be a positive Radon measure with compact support $V \neq \{0\}$ and let $\Omega = \mathbb{R}^N$ with $N \geq 2$. Let $\sigma = 0$ and*

$$K : L^1(\mathbb{R}^N \times V) \ni \varphi \mapsto \int_V \varphi(x, v')d\mu(v').$$

Then there exists a sequence $t_n \to 0$ such that $t \mapsto R_1(t)$ is not norm continuous at t_n for all n.

Proof. If (for some $T > 0$) $]0, T] \ni t \mapsto R_1(t)$ is norm continuous, then $]0, T] \ni t \mapsto U_1(t)$ would be also norm continuous [4, Lemma 2.3, p. 16]. Thus it suffices to prove that $0 < t \mapsto U_1(t)$ is *not* norm continuous. Let $\bar{t} > 0$. We recall that

$$U_1(t) = \int_0^t U(t-s)KU(s)ds.$$

To show that $0 < t \mapsto U_1(t)$ is not norm continuous at $\bar{t}$ it suffices to show that

$$0 < t \mapsto \widetilde{U_1}(t) := \int_0^{\bar{t}} U(t-s)KU(s)ds$$

is not norm continuous at $\bar{t}$. Let

$$L_{\overline{v}} := \{\alpha\overline{v}; \quad \alpha \in \mathbb{R}\}$$

be the line with direction $\overline{v} \in S^{N-1}$. Without loss of generality, we may assume that

$$\mu\{L_{\overline{v}}\} < \|\mu\|. \tag{2.26}$$

Indeed, if for some $\overline{v} \in S^{N-1}$ $\mu\{L_{\overline{v}}\} = \|\mu\|$, then for *all* $\overline{\overline{v}} \in S^{N-1}$ with $\overline{\overline{v}} \neq \overline{v}$ we have $L_{\overline{v}} \cap L_{\overline{\overline{v}}} = \{0\}$ and

$$\begin{aligned} \|\mu\| &\geq \mu\{L_{\overline{v}} \cup L_{\overline{\overline{v}}}\} \\ &= \mu\{L_{\overline{v}}\} + \mu\{L_{\overline{\overline{v}}}\} - \mu\{0\} = \|\mu\| + \mu\{L_{\overline{\overline{v}}}\} - \mu\{0\} \end{aligned}$$

so

$$\mu\{L_{\overline{\overline{v}}}\} \leq \mu\{0\} < \|\mu\|$$

since μ is *not* supported by $\{0\}$. Let $(f_j)_j \subset L^1(\mathbb{R} \times V)$ be a normalized sequence converging in the weak star topology of measures to the Dirac mass $\delta_{(0,\overline{v})} = \delta_{x=0} \otimes \delta_{v=\overline{v}}$. It is clear that

$$\begin{aligned} \|\widetilde{U_1}(t) - \widetilde{U_1}(\bar{t})\| &\geq \sup_{j \in \mathbb{N}} \|\widetilde{U_1}(t)f_j - \widetilde{U_1}(\bar{t})f_j\| \\ &= \sup_{j \in \mathbb{N}} \sup_{\{\varphi \in \mathcal{C}_c(\mathbb{R}^N \times V); \quad \|\varphi\|_\infty = 1\}} \langle \widetilde{U_1}(t)f_j - \widetilde{U_1}(\bar{t})f_j, \varphi \rangle \end{aligned}$$

$$\geq \sup_{\{\varphi\in\mathcal{C}_c(\mathbb{R}^N\times V);\quad \|\varphi\|_\infty=1\}} \overline{\lim}_{j\to\infty}\langle \widetilde{U_1}(t)f_j-\widetilde{U_1}(\bar{t})f_j,\varphi\rangle,$$

where $\mathcal{C}_c(\mathbb{R}^N\times V)$ stands for the space of continuous functions with compact supports. On the other hand $\langle \widetilde{U_1}(t)f_j,\varphi\rangle$ is equal to

$$\begin{aligned}
&\int_{\mathbb{R}^N\times V} dx d\mu(v)\varphi(x,v)\int_0^{\bar{t}} ds\int_V f_j(x-(t-s)v-sv',v')d\mu(v') \\
=&\int_V d\mu(v)\int_V d\mu(v')\int_0^{\bar{t}} ds\int_{\mathbb{R}^N} f_j(y,v')\varphi(y+(t-s)v+sv',v)dy \\
=&\int_{\mathbb{R}^N\times V} dy d\mu(v')f_j(y,v')\int_V d\mu(v)\int_0^{\bar{t}}\varphi(y+(t-s)v+sv',v)ds \\
\to&\int_V d\mu(v)\int_0^{\bar{t}}\varphi((t-s)v+s\overline{v},v)ds
\end{aligned}$$

as $j\to\infty$ so

$$\|\widetilde{U_1}(t)-\widetilde{U_1}(\bar{t})\|\geq \sup_{\{\varphi\in\mathcal{C}_c(\mathbb{R}^N\times V);\quad \|\varphi\|_\infty=1\}}\int_V d\mu(v)\int_0^{\bar{t}}\varphi((t-s)v+s\overline{v},v)-\varphi((\bar{t}-s)v+s\overline{v},v)ds.$$

Our goal, now, is to prove that for every $t\neq\bar{t}$ the last supremum is bounded away by a positive constant *independent* of t. Let

$$\Gamma_{t,v}:=\{tv+s(\overline{v}-v);\quad s\in[0,\bar{t}]\}$$

be a segment starting at tv with direction $\overline{v}-v$. For $t\neq\bar{t}$ and $v\notin L_{\overline{v}}$ the segments $\Gamma_{t,v}$ and $\Gamma_{\bar{t},v}$ are *disjoint*. Let

$$L_{t,v}:=\{tv+s(\overline{v}-v);\quad s\in\mathbb{R}\}$$

be the line passing through tv and with direction $\overline{v}-v$ (it contains $\Gamma_{t,v}$). Note that $L_{t,v}$ and $L_{\bar{t},v}$ are *disjoint* if $v\notin L_{\overline{v}}$ and are identical otherwise. Define the function φ^t on $\mathbb{R}^N\times(V\backslash L_{\overline{v}})$ by:

$$\varphi^t(x,v)=\frac{d(x,L_{\bar{t},v})}{d(x,L_{t,v})+d(x,L_{\bar{t},v})}$$

where $d(x,L_{t,v})=\inf_{y\in L_{t,v}}|x-y|=\inf_{s\in\mathbb{R}}|x-tv-s(\overline{v}-v)|$ is given by

$$d(x,L_{t,v})=|x-tv-\langle x-tv,\overline{v}-v\rangle|\overline{v}-v|^{-2}(\overline{v}-v)|$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^N$. Then φ^t is continuous on $\mathbb{R}^N\times(V\backslash L_{\overline{v}})$ and

$$\varphi^t(x,v)=\begin{cases}1 & \text{for}\quad x\in\Gamma_{t,v}\\ 0 & \text{for}\quad x\in\Gamma_{\bar{t},v}.\end{cases}\tag{2.27}$$

Let $\theta_\varepsilon : V \to \mathbb{R}$ be a continuous function satisfying $0 \le \theta_\varepsilon(v) \le 1$, $\theta_\varepsilon(v) = 0$ if $v \in C^1_\varepsilon$ and $\theta_\varepsilon(v) = 1$ if $v \notin C^2_\varepsilon$, where

$$C^1_\varepsilon = V \cap \{v \in \mathbb{R}^N;\;\; d(v, L_{\overline{v}}) \le \frac{\varepsilon}{2}\}$$

and

$$C^2_\varepsilon = V \cap \{v \in \mathbb{R}^N;\;\; d(v, L_{\overline{v}}) \le \varepsilon\}.$$

The function $\theta_\varepsilon \varphi^t : \mathbb{R}^N \times V \ni (x, v) \mapsto \varphi^t(x, v)\theta_\varepsilon(v)$ is continuous in $\mathbb{R}^N \times V$. Let $\psi \in \mathcal{C}_c(\mathbb{R}^N)$ be a function verifying $0 \le \psi \le 1$ and $\psi = 1$ on the compact set $\overline{\bigcup_{(t,v)\in\Xi} \Gamma_{t,v}}$, where $\Xi = [\overline{t} - \delta, \overline{t} + \delta] \times V$ (for some fixed $\delta > 0$). Using (2.27), the function

$$\phi^t_\varepsilon : \mathbb{R}^N \times V \ni (x, v) \mapsto \varphi^t(x, v)\theta_\varepsilon(v)\psi(x),$$

which is in $\mathcal{C}_c(\mathbb{R}^N \times V)$ because ψ has compact support in $\mathbb{R}^N$ and V is compact, satisfies

$$\begin{aligned}
& \int_V \int_0^{\overline{t}} \phi^t_\varepsilon((t-s)v + s\overline{v}, v) - \phi^t_\varepsilon((\overline{t}-s)v + s\overline{v}, v)\, ds\, d\mu(v) \\
= & \int_V \int_0^{\overline{t}} \phi^t_\varepsilon((t-s)v + s\overline{v}, v)\, ds\, d\mu(v) \\
\ge & \int_{V\setminus C^2_\varepsilon} \int_0^{\overline{t}} \phi^t_\varepsilon((t-s)v + s\overline{v}, v)\, ds\, d\mu(v) \\
= & \; \overline{t}\mu\{V\setminus C^2_\varepsilon\}.
\end{aligned}$$

Finally, since $\mu\{V\setminus C^2_\varepsilon\} \to \mu\{V\setminus L_{\overline{v}}\}$ as $\varepsilon \to 0$, it follows from (2.26) that

$$\|\widetilde{U_1}(t) - \widetilde{U_1}(\overline{t})\| \ge \overline{t}\mu\{V\setminus L_{\overline{v}}\} > 0$$

for all $t \ne \overline{t}$. ■

2.2 The dimension one

The one dimensional theory ($N = 1$) is very different. We have:

Theorem 2.11. *Let μ be a positive Radon measure on $\mathbb{R}$ satisfying*

$$\sup_{v'\in\mathbb{R}} \mu\left\{\left[v' - \varepsilon, v' + \varepsilon\right]\right\} \to 0 \text{ as } \varepsilon \to 0. \tag{2.28}$$

We assume that (H3) is satisfied. Then

$$0 < t \mapsto R_1(t) \in \mathcal{L}(L^1(\mathbb{R} \times \mathbb{R}))$$

is norm continuous.

Proof. We recall that $0 \le t \mapsto R_1(t)$ is norm continuous if and only if

$$0 \le t \mapsto U_1(t) = \int_0^t U(t-s)KU(s)ds$$

is so [4, Theorem 2.7, p. 18]. By density arguments we may suppose that K is of the form

$$K : L^1(\mathbb{R} \times \mathbb{R}) \ni \varphi \mapsto \int_{\mathbb{R}} \varphi(x, v')f(v)g(v')d\mu(v') \in L^1(\mathbb{R} \times \mathbb{R}),$$

where $g(\cdot) \in L^\infty(\mathbb{R})$ and $f(\cdot)$ is continuous with *compact support*. We have

$$\begin{aligned} U_1(t)\varphi &= \int_0^t U(t-s)KU(s)\varphi ds \\ &= \int_0^t \int_{\mathbb{R}} f(v)e^{-(t-s)\sigma(v)-s\sigma(v')}\varphi(x-(t-s)v-sv', v')g(v')dsd\mu(v') \\ &= \int_{-\infty}^{v} \int_{x-tv}^{x-tv'} \varphi(y, v')\Theta(t, v, v', x, y)(v-v')^{-1}dyd\mu(v') \\ &\quad + \int_{v}^{+\infty} \int_{x-tv}^{x-tv'} \varphi(y, v')\Theta(t, v, v', x, y)(v-v')^{-1}dyd\mu(v') \\ &=: O_1(t)\varphi + O_2(t)\varphi, \end{aligned}$$

where

$$\Theta(t, v, v', x, y) = f(v)g(v')e^{-(x-y-tv')(v-v')^{-1}\sigma(v)}e^{-(y-x+tv)(v-v')^{-1}\sigma(v')}.$$

Let us show that both $0 < t \mapsto O_1(t)$ and $0 < t \mapsto O_2(t)$ are norm continuous. We restrict ourselves for instance to $0 < t \mapsto O_1(t)$ since the same argument holds for $0 < t \mapsto O_2(t)$. Note that

$$O_1(t)\varphi = \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(y, v')\Theta(t, v, v', x, y)E(t, v, v', x, y)dyd\mu(v')$$

where

$$E(t, v, v', x, y) = \chi_{\{v'<v\}}\chi_{\{y+tv' \le x \le y+tv\}}|v-v'|^{-1}.$$

Let

$$O_1^\varepsilon(t) : \varphi \mapsto \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(y, v')\Theta(t, v, v', x, y)E_\varepsilon(t, v, v', x, y)dyd\mu(v')$$

where $E_\varepsilon(t, v, v', x, y) = E(t, v, v', x, y)\chi_{\{\varepsilon \le |v-v'|\}}$. We are going to show that

$$\|O_1(t) - O_1^\varepsilon(t)\| \to 0 \text{ as } \varepsilon \to 0$$

uniformly on $t \in [0, T]$. Since

$$|\Theta(t, v, v', x, y)| \le e^{2T\|\sigma\|_\infty}\|f\|_\infty\|g\|_\infty =: C$$

when $t \in [0,T]$ and $(t,v,v',x,y) \in \text{Supp } E$, we have

$$\begin{aligned}
&\|O_1(t)\varphi - O_1^\varepsilon(t)\varphi\| \\
= \ & \int_{\mathbb{R}} d\mu(v) \int_{\mathbb{R}} dx \Big| \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(y,v')\Theta(t,v,v',x,y)[E(t,v,v',x,y) - E_\varepsilon(t,v,v',x,y)] dy d\mu(v') \Big| \\
\le \ & C \int_{\mathbb{R}} d\mu(v) \int_{\mathbb{R}} dx \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(y,v')| E(t,v,v',x,y) \chi_{\{|v-v'|<\varepsilon\}} dy d\mu(v') \\
\le \ & C \int_{\mathbb{R}} d\mu(v') \int_{\mathbb{R}} |\varphi(y,v')| dy \int_{\mathbb{R}} \chi_{\{|v-v'|<\varepsilon\}} d\mu(v) \int_{\mathbb{R}} E(t,v,v',x,y) dx.
\end{aligned}$$

Now $\int_{\mathbb{R}} E(t,v,v',x,y)dx = |v-v'|^{-1} \int_{y+tv'}^{y+tv} dx = t$ implies

$$\begin{aligned}
\|O_1(t)\varphi - O_1^\varepsilon(t)\varphi\| &\le Ct \int_{\mathbb{R}} d\mu(v') \int_{\mathbb{R}} |\varphi(y,v')| \, dy \int_{\mathbb{R}} \chi_{\{|v-v'|<\varepsilon\}} d\mu(v) \\
&\le CT \sup_{v' \in \mathbb{R}} \mu \left\{ [v'-\varepsilon, v'+\varepsilon] \right\} \|\varphi\|,
\end{aligned}$$

which shows, by using (2.28), that $\|O_1(t) - O_1^\varepsilon(t)\| \to 0$ as $\varepsilon \to 0$ uniformly on $t \in [0,T]$. Let us show that $0 < t \mapsto O_1^\varepsilon(t)$ is norm continuous. Let $\bar{t} > 0$, we have

$$\begin{aligned}
&\left\|O_1^\varepsilon(t)\varphi - O_1^\varepsilon(\bar{t})\varphi\right\| \\
= \ & \int_{\mathbb{R}} d\mu(v) \int_{\mathbb{R}} dx \Big| \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(y,v')[\Theta(t,v,v',x,y)E_\varepsilon(t,v,v',x,y) \\
& \qquad - \Theta(\bar{t},v,v',x,y)E_\varepsilon(\bar{t},v,v',x,y)] dy d\mu(v') \Big| \\
\le \ & \int_{\mathbb{R}} d\mu(v) \int_{\mathbb{R}} dx \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(y,v')| \left|\Theta(t,v,v',x,y) - \Theta(\bar{t},v,v',x,y)\right| \left|E_\varepsilon(t,v,v',x,y)\right| dy d\mu(v') \\
+ \ & \int_{\mathbb{R}} d\mu(v) \int_{\mathbb{R}} dx \int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(y,v')| \left|\Theta(\bar{t},v,v',x,y)\right| \left|E_\varepsilon(t,v,v',x,y) - E_\varepsilon(\bar{t},v,v',x,y)\right| dy d\mu(v') \\
=: \ & I_1 + I_2.
\end{aligned}$$

Since $\int_{\mathbb{R}} E_\varepsilon(t,v,v',x,y)dx \le t$ and

$$\begin{aligned}
&\left|\Theta(t,v,v',x,y) - \Theta(\bar{t},v,v',x,y)\right| \\
\le \ & \frac{C_1}{|v-v'|}(|v'|\sigma(v) + |v|\sigma(v'))|f(v)||t-\bar{t}| \le \Big[1 + \frac{2d}{\varepsilon}\Big] C_1 \|\sigma\|_\infty |f(v)||t-\bar{t}|
\end{aligned}$$

for $t \in [0,T]$ and $(t,v,v',x,y) \in \text{Supp } E_\varepsilon$, where $d = \max\{|v|;\ \ v \in \text{ Supp } f\}$ and $C_1 = \|g\|_\infty$, we get

$$I_1 \le T\Big[1 + \frac{2d}{\varepsilon}\Big] C_1 \|\sigma\|_\infty \|f\|_1 |t-\bar{t}| \|\varphi\|. \tag{2.29}$$

On the other hand, on can establish that

$$\int_{\mathbb{R}} \left|E_\varepsilon(t,v,v',x,y) - E_\varepsilon(\bar{t},v,v',x,y)\right| dx \le \Big[1 + \frac{2d}{\varepsilon}\Big] \left|t-\bar{t}\right|,$$

so

$$I_2 \leq C_2\Big[1 + \frac{2d}{\varepsilon}\Big]\|f\|_1|t-\bar{t}|\|\varphi\|, \tag{2.30}$$

where $C_2 = e^{2T\|\sigma\|_\infty}\|g\|_\infty$. Combining (2.29) and (2.30)

$$\big\|O_1^\varepsilon(t)\varphi - O_1^\varepsilon(\bar{t})\big\| \quad \leq \quad \Big[TC_1\|\sigma\|_\infty + C_2\Big]\Big[1+\frac{2d}{\varepsilon}\Big]\|f\|_1|t-\bar{t}| \to 0$$

as $t \to \bar{t}$. This ends the proof. ■

Remark 2.12. Assumption (2.28) is fulfilled by the Lebesgue measure on $\mathbb{R}$. We note that if μ has a compact support, then (2.28) is equivalent to the assumption that μ is diffuse, i.e. $\mu\{v\} = 0$ for all $v \in \mathbb{R}$.

We now show the *optimality* of Theorem 2.11.

Theorem 2.13. *Let μ be a positive Radon measure on $\mathbb{R}$ with compact support V which is not diffuse i.e. $\mu\{\overline{v}\} \neq 0$ for some $\overline{v} \in V$. We suppose that $\sigma = 0$ and that*

$$K : L^1(\mathbb{R}\times V) \ni \varphi \mapsto \int\limits_V \varphi(x,v)d\mu(v).$$

Then there exists a sequence $t_n \to 0$ such that $t \mapsto R_1(t)$ is not norm continuous at t_n for all n.

Proof. As in the proof of Theorem 2.10 we are led to show that

$$\sup_{\{\varphi\in\mathcal{C}_c(\mathbb{R}\times V);\ \ \|\varphi\|_\infty=1\}} \int\limits_V \int\limits_0^{\bar{t}} \varphi((t-s)v+s\overline{v},v) - \varphi((\bar{t}-s)v+s\overline{v},v)dsd\mu(v) \nrightarrow 0$$

as $t \to \bar{t}$, where $\mathcal{C}_c(\mathbb{R}\times V)$ is the space of continuous functions with compact supports. To this end, let $(\psi_n)_n \subset \mathcal{C}(V)$ (with $\|\psi_n\|_\infty = 1$) be a sequence converging pointwise to $\chi_{\{\overline{v}\}}$. By the dominated convergence theorem, for all $\phi \in \mathcal{C}_c(\mathbb{R})$ we have

$$\int\limits_V \int\limits_0^{\bar{t}} \phi((t-s)v+s\overline{v})\psi_n(v) - \phi((\bar{t}-s)v+s\overline{v})\psi_n(v)dsd\mu(v) \to \bar{t}\mu\{\overline{v}\}[\phi(t\overline{v}) - \phi(\bar{t}\overline{v})]$$

as $n \to \infty$. Finally

$$\begin{aligned}&\sup_{\{\varphi\in\mathcal{C}_c(\mathbb{R}\times V);\ \ \|\varphi\|_\infty=1\}} \int\limits_V \int\limits_0^{\bar{t}} \varphi((t-s)v+s\overline{v},v) - \varphi((\bar{t}-s)v+s\overline{v},v)dsd\mu(v)\\ &\geq \bar{t}\mu\{\overline{v}\} \sup_{\{\phi\in\mathcal{C}_c(\mathbb{R});\ \ \|\phi\|_\infty=1\}} [\phi(t\overline{v}) - \phi(\bar{t}\overline{v})].\end{aligned}$$

Choosing a function $\phi \in \mathcal{C}_c(\mathbb{R})$ such that $0 \leq \phi(x) \leq 1$, $\phi(t\overline{v}) = 1$ and $\phi(\bar{t}\overline{v}) = 0$ we get

$$\sup_{\{\phi\in\mathcal{C}_c(\mathbb{R});\ \ \|\phi\|_\infty=1\}} \bar{t}\mu\{\overline{v}\}[\phi(t\overline{v}) - \phi(\bar{t}\overline{v})] \geq \bar{t}\mu\{\overline{v}\}$$

for all $t \neq \bar{t}$. ■

3 On Dyson-Phillips expansions on arbitrary domains

In this section, we recall a continuity result which holds in arbitrary domains Ω for *not necessarily* space homogeneous scattering kernels [4, Proposition 4.2, p.77]. The usual continuous models are covered by this result but the multigroup models *are not*.

Theorem 3.1. **[4, Proposition 4.2, p.77]** *Let $\Omega \subset \mathbb{R}^N$ be an open set and let (H1)-(H2) be satisfied. Let*

$$t \mapsto \sigma(tv) \text{ be continuous for each } v \in V. \tag{3.1}$$

Let $d\mu(v) = d\gamma(\rho) \otimes d\beta(\omega)$ where $v = \rho\omega$, $\omega \in S^{N-1}$, β is a Radon measure on S^{N-1} and $d\gamma(\rho) = h(\rho)d\rho$ ($h \in L^1_{loc}(0,\infty)$). Then $0 < t \mapsto R_2(t)$ is norm continuous.

Remark 3.2. Actually, Theorem 3.1 holds in L^p spaces ($1 \leq p < \infty$). This can be shown by density arguments and interpolation. This answers partly Problem 3 in [7] about the relevance of the convexity assumption on Ω.

4 Spectral mapping theorems

Before giving the main results of this section we state some preliminary results. The first one provides a description of the spectrum of the streaming semigroup and its generator.

Lemma 4.1. (i) *Let one of the following assumptions be satisfied:*

(A1) *$\sigma(v) = \sigma(-v)$, the hyperplanes have zero μ-measure and Ω is included in a half-space.*

(A2) *Ω is the exterior domain of a bounded and open subset O (i.e. $\Omega = \mathbb{R}^N \backslash D$ with $D = \overline{O}$).*

Then

$$\sigma(T) = \sigma_{ap}(T) = \{\lambda; \quad Re\lambda \leq -\lambda^*\}$$

and

$$\sigma(U(t)) = \sigma_{ap}(U(t)) = \sigma_{crit}(U(t)) = \left\{\mu; \quad |\mu| \leq e^{-t\lambda^*}\right\},$$

where $\lambda^* = \lim_{t\to\infty} \inf_{\{(x,v);t<\tau(x,v)\}} \sigma(v)$.

(ii) *If $\Omega = \mathbb{R}^N$ then the real spectrum of T i.e. $\sigma(T) \cap \mathbb{R}$ is equal to the essential range of $-\sigma$ and*

$$\sigma(T) = \sigma_{ap}(T) = (\sigma(T) \cap \mathbb{R}) + i\mathbb{R},$$

while

$$\sigma(U(t)) = \sigma_{ap}(U(t)) = \sigma_{crit}(U(t)) = e^{t\sigma(T)}.$$

Proof. (i) First we deal with the approximate spectrum. Following [6] the streaming semigroup $(U(t))_{t\geq 0}$ can be decomposed into three positive semigroups (with independent dynamics). To this end, define the sets

$$\Xi_1 := \{(x,v) \in \Omega \times \mathbb{R}^N; \quad \tau(x,-v) < \infty\},$$

$$\Xi_2 := \{(x,v) \in \Omega \times \mathbb{R}^N; \quad \tau(x,-v) = \infty, \ \tau(x,v) < \infty\}$$

and

$$\Xi_3 := \{(x,v) \in \Omega \times \mathbb{R}^N; \quad \tau(x,-v) = \infty, \ \tau(x,v) = \infty\}\}.$$

Identifying $L^p(\Xi_i)$ $(i = 1, 2, 3)$ and the (closed) subspace of $L^p(\Omega \times \mathbb{R}^N)$

$$\{f \in L^p(\Omega \times \mathbb{R}^N); \quad f(x,v) = 0 \ a.e. \text{ on } \Omega \times \mathbb{R}^N - \Xi_i\}.$$

Then the subspaces $L^p(\Xi_i)$ $(i = 1, 2, 3)$ are invariant under $(U(t))_{t\geq 0}$ and we have

$$L^p(\Omega \times \mathbb{R}^N) = L^p(\Xi_1) \oplus L^p(\Xi_2) \oplus L^p(\Xi_3),$$

$$\sigma(U(t)) = \sigma(U_1(t)) \cup \sigma(U_2(t)) \cup \sigma(U_3(t)) \quad \text{and} \quad \sigma(T) = \sigma(T_1) \cup \sigma(T_2) \cup \sigma(T_3),$$

where $U_i(t) = U(t)_{|L^p(\Xi_i)}$ and T_i is the generator of $(U_i(t))_{t\geq 0}$ $(i = 1, 2, 3)$. Moreover by [6, Theorem 7] if $dx \otimes d\mu(\Xi_1) > 0$, then the (approximate) spectrum of the semigroup $(U_1(t))_{t\geq 0}$ and its generator T_1 are given by

$$\sigma(U_1(t)) = \sigma_{ap}(U_1(t)) = \{\mu; \quad |\mu| \leq e^{-\lambda_1^* t}\} \quad \text{and} \quad \sigma(T_1) = \sigma_{ap}(T_1) = \{\lambda; \quad Re\lambda \leq -\lambda_1^*\}$$

where

$$\lambda_1^* = \lim_{t\to\infty} \inf_{\{\tau(x,-v)<\infty,\ t<\tau(x,v)\}} \sigma(v).$$

On the other hand, by [6] again we have

$$s(T) = \omega_0(U(\cdot)) = -\lambda^* = -\lim_{t\to\infty} \inf_{\{(x,v);t<\tau(x,v)\}} \sigma(v).$$

Hence to prove the part concerning the approximate spectrum it suffices to show that Assumption **(A1)** or **(A2)** implies $\lambda_1^* = \lambda^*$. First, since

$$E_1(t) := \{(x, v); \quad \tau(x, -v) < \infty,\ t < \tau(x, v)\} \subset \{(x, v); \quad t < \tau(x, v)\} =: E(t)$$

it follows that $\lambda^* \leq \lambda_1^*$. Suppose now that (**A1**) holds. Let us prove the following implication

$$(x, v) \in E(t) \Rightarrow (x, v) \in E_1(t) \text{ or } (x - tv, -v) \in E_1(t).$$

Indeed, if $(x, v) \notin E_1(t)$, then since Ω is included in a half-space $\tau(x, -(-v)) < \infty$, and so $\tau(x - tv, -(-v)) < \infty$. The last inequality implies $\tau(x - tv, -v) = \infty$ because Ω is included in a half-space, so that $t < \tau(x - tv, -v)$ and $(x - tv, -v) \in C_1(t)$. Consequently, to each $(x, v) \in E(t)$ we can associate an element $(x, v) \in E_1(t)$ or $(x - tv, -v) \in E_1(t)$ with

$$\sigma(x - tv, -v) = \sigma(x, v) = \sigma(v).$$

Thus

$$\lambda_1^* \leq \lambda^*.$$

Under Assumption **(A2)**, for each $w \in V$ there exist a cone C_w containing w and a ball $B(x_w, \varepsilon_w) \subset \Omega$ such that

$$B(x_w, \varepsilon_w) \times C_w \subset \{(x, v) \in \Omega \times \mathbb{R}^N; \quad \tau(x, -v) < \infty,\ \tau(x, v) = \infty\} \subset E_1(t).$$

Hence

$$\lambda_1^* \leq \inf_{\cup_{w\in V} B(x_w, \varepsilon_w) \times C_w} \sigma(w) = \inf \sigma(v).$$

On the other hand it is clear that

$$\inf \sigma(v) \leq \lambda^* \leq \lambda_1^*,$$

which gives the equality $\lambda_1^* = \lambda^*$. Finally, the part concerning the critical spectrum is a consequence of Theorem 1.4.

(ii) When $\Omega = \mathbb{R}^N$ and the collision frequency is homogeneous, according to [6, Theorem 10], the real spectrum of T is equal to the essential range of $-\sigma(\cdot)$. By [6, Theorem 9] $\sigma(T) = (\sigma(T) \cap \mathbb{R}) + i\mathbb{R}$ and $\sigma(U(t)) = e^{t\sigma(T)}$. The proof of [6, Theorem 10] shows also that $\sigma_{ap}(T) = \sigma(T)$. For the critical spectrum we use again Theorem 1.4. ■

We recall

Lemma 4.2. **[7, lemma 7]** *We assume that the measure μ satisfies*

$$\int_D e^{iz.v} d\mu(v) \to 0 \ \ as \ \ |z| \to \infty \tag{4.1}$$

for all Borel set $D \subset \mathbb{R}^N$ with $\mu(D) < \infty$ and that (H1)-(H2) are satisfied. Then

$$\sigma_{ap}(T) \subset \sigma_{ap}(T + K).$$

Remark 4.3. We know [7, Lemma 5] that under Assumption (4.1), the affine hyperplanes have zero μ-measure.

We are now in position to give the main results in the whole space.

Theorem 4.4. *Let $\Omega = \mathbb{R}^N$. We assume that the scattering kernel is space homogeneous and that (H3) is satisfied.*

(a) Let $N \geq 2$. If (2.1) is satisfied then

$$\sigma(V(t)) \cap \{\mu; \ \ |\mu| < e^{-t\lambda^{**}} \ or \ |\mu| > e^{-t\lambda^*}\} = e^{t(\sigma(T+K)\cap\{\lambda; \ \ Re\lambda<-\lambda^{**} \ or \ Re\lambda>-\lambda^*\})},$$

where $\lambda^* = ess \inf \sigma(\cdot)$ and $\lambda^{**} = ess \sup \sigma(\cdot)$. Moreover, if (4.1) is satisfied and if the essential range of $\sigma(\cdot)$ is connected then

$$\sigma(V(t)) = e^{t\sigma(T+K)}.$$

(b) If $N = 1$ and if μ satisfy (2.28) and (4.1) then

$$\sigma(V(t)) = e^{t\sigma(T+K)}.$$

Proof. From Theorems 1.2 and 2.1 it follows that $\omega_{crit}(V(\cdot)) = \omega_{crit}(U(\cdot))$. Using Lemma 4.1(ii) we get $\omega_{crit}(V(\cdot)) = -\lambda^*$ and, by Theorem 1.1 (c),

$$\sigma(V(t)) \cap \{\mu; \ \ |\mu| > e^{-t\lambda^*}\} = e^{t(\sigma(T+K)\cap\{\lambda; \ \ Re\lambda>-\lambda^*\})}. \tag{4.2}$$

On the other hand, when ($\Omega = \mathbb{R}^N$), the semigroup $(U(t))_{t\geq 0}$ can be extended to a positive group with

$$U(-t) : \varphi \mapsto e^{t\sigma(v)}\varphi(x + tv) \ (t \geq 0).$$

The latter is generated by $-T$ and its growth bound is λ^{**}. Similarly $(V(t))_{t\geq 0}$ can be extended to a group with $(V(-t))_{t\geq 0}$ generated by $-T-K$ and can be represented by Dyson-Phillips series. We can show by the same arguments as in Theorem 2.1 that some remainder term depends continuously on t in operator norm. Thus,

$$\omega_{crit}(V(-\cdot)) = \omega_{crit}(U(-\cdot)) = \lambda^{**}$$

and

$$\sigma(V(-t)) \cap \left\{\mu; \quad |\mu| > e^{t\lambda^{**}}\right\} = e^{t(\sigma(-T-K)\cap\{\lambda; \quad Re\lambda>\lambda^{**}\})}.$$

On the other hand $\mu \in \sigma(V(-t))$ if and only if $\mu^{-1} \in \sigma(V(t))$, so

$$\sigma(V(t)) \cap \left\{\mu; \quad |\mu| < e^{-t\lambda^{**}}\right\} = e^{t(\sigma(T+K)\cap\{\lambda; \quad Re\lambda<-\lambda^{**}\})}$$

and this *complements* (4.2). We note that Lemmas 4.1(ii) and 4.2, and the connectedness of the essential range of $\sigma(\cdot)$ imply

$$\sigma(V(t)) \cap \{\mu; \quad e^{-t\lambda^{**}} \leq |\mu| \leq e^{-t\lambda^{*}}\} = e^{t(\sigma(T+K)\cap\{\lambda; \quad -\lambda^{**}\leq Re\lambda\leq -\lambda^{*}\})},$$

which ends the proof of (a). We now prove the item (b). Since $0 < t \mapsto R_1(t)$ is norm continuous (Theorem 2.11), Theorem 1.3 implies that

$$\sigma_{crit}(V(t)) = \sigma_{crit}(U(t)) \text{ for all } t \geq 0.$$

Combining this with Lemmas 4.1(ii) and 4.2 we get

$$\sigma_{crit}(V(t)) \subset e^{t\sigma(T+K)}$$

which ends the proof. ■

Remark 4.5. For the special case $d\mu(v) = dv$, the Lebesgue measure on $\mathbb{R}^N$, the item (a) of Theorem 4.4 holds for not necessarily space homogeneous scattering kernels. This can be shown by using Theorem 3.1.

Before dealing with *general* spatial domains we need one more preliminary lemma.

Lemma 4.6. *Let $d\mu(v) = d\gamma(\rho) \otimes d\beta(\omega)$ where $v = \rho\omega$, $\omega \in S^{N-1}$, γ is a Radon measure on $(0, \infty)$ absolutely continuous with respect to Lebesgue measure and β is a Radon measure on S^{N-1} satisfying*

$$\beta\{\omega \in S^{N-1}; \quad \omega \cdot \omega_0 = 0\} = 0 \quad \textit{for all } \omega_0 \in S^{N-1}.$$

Then μ satisfies (4.1).

Proof. A simple compactness argument shows that

$$\lim_{\varepsilon\to 0} \sup_{\omega_0 \in S^{N-1}} \beta\{\omega \in S^{N-1}; \quad |\omega \cdot \omega_0| \leq \varepsilon\} = 0. \tag{4.3}$$

We have to show that

$$\int_D e^{iz\cdot v} d\mu(v) \to 0 \text{ as } |z| \to \infty$$

for all Borel set $D \subset \mathbb{R}^N$ with $\mu\{D\} < \infty$. The last statement is equivalent to show that

$$\int_{\mathbb{R}^{\mathbb{N}}} h(v) e^{iz\cdot v} d\mu(v) \to 0 \text{ as } |z| \to \infty$$

for all $h \in L^1(\mathbb{R}^N;\mu)$. Since the continuous functions with compact support are dense in $L^1(\mathbb{R}^N;\mu)$ it suffices to deal with a continuous function h with compact support, say with support included in $B(0,R)$, the ball centred at 0 with radius R.

$$\begin{aligned}
\int_{\mathbb{R}^{\mathbb{N}}} h(v) e^{iz\cdot v} d\mu(v) &= \int_{S^{N-1}} d\beta(\omega) \int_0^R h(\rho,\omega) e^{i\rho z\cdot\omega} d\gamma(\rho) \\
&= \int_{S^{N-1}} d\beta(\omega) \int_0^R h(\rho,\omega) e^{i\rho|z|\omega'\cdot\omega} d\gamma(\rho) \\
&= \int_{\{\omega\in S^{N-1};\ |\omega'\cdot\omega|\le\varepsilon\}} d\beta(\omega) \int_0^R h(\rho,\omega) e^{i\rho|z|\omega'\cdot\omega} d\gamma(\rho) \\
&\quad + \int_{\{\omega\in S^{N-1};\ |\omega'\cdot\omega|>\varepsilon\}} d\beta(\omega) \int_0^R h(\rho,\omega) e^{i\rho|z|\omega'\cdot\omega} d\gamma(\rho) \\
&=: I_1 + I_2,
\end{aligned}$$

where $v = \rho\omega$ and $z = |z|\omega'$. From (4.3) it is easy to see that I_1 is arbitrarily small for ε small enough. We fix ε small enough and consider the second term. Since h is continuous and γ is absolutely continuous with respect to Lebesgue measure,

$$\int_0^R h(\rho,\omega) e^{i\rho|z|\omega'\cdot\omega} d\gamma(\rho) \to 0 \text{ as } |z| \to \infty$$

uniformly in ω and ω' such that $|\omega\cdot\omega'| > \varepsilon$ so $I_2 \to 0$ as $|z| \to \infty$. ■

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^N$ be an open set and let (H1)-(H2) be satisfied. Let $d\mu(v) = d\gamma(\rho) \otimes d\beta(\omega)$ where γ is a Radon measure on $(0,\infty)$ absolutely continuous with respect to Lebesgue measure and β is a Radon measure on S^{N-1}. We assume that (3.1) is satisfied. Then*

$$\sigma(V(t)) \cap \{\mu;\ \ |\mu| > e^{-t\lambda^*}\} = e^{t(\sigma(T+K)\cap\{\lambda;\ \ Re\lambda > -\lambda^*\})}. \tag{4.4}$$

Moreover, if β is such that $\beta\{\omega \in S^{N-1};\ \ \omega\cdot\omega_0 = 0\} = 0$ for all $\omega_0 \in S^{N-1}$ and if Ω satisfies one of Assumptions (A1) or (A2), then

$$\sigma(V(t)) = e^{t\sigma(T+K)} \cup \{0\}.$$

Proof. As in the proof of item *(a)* of Theorem 4.4, the use of Theorems 3.1, 1.2 and 1.1, and Lemma 4.1(i) gives (4.4). On the other hand, if Ω satisfies one of Assumptions (A1) or (A2) then by Lemmas 4.1(i), 4.2 and 4.6 and the inclusion $e^{t\sigma_{ap}(T+K)} \subset \sigma_{ap}(V(t))$ we obtain

$$\{\mu;\ \ |\mu| \le e^{-t\lambda^*}\} \subset e^{t\sigma(T+K)} \subset \sigma(V(t))$$

which ends the proof. ■

Remark 4.8. In bounded geometries, the spectral mapping theorems are consequences of compactness results obtained under the assumption that the collision operator is "weakly compact with respect to velocities", where the weak compactness is "collective" with respect to the space variable [3, 5]. This suggests that (H1)(H2) could probably be replaced by a collective weak compactness assumption.

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