A resolvent approach to the stability of essential and critical spectra of perturbed C_0-semigroups on Hilbert spaces with applications to transport theory

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Abstract

We give sufficient conditions in terms of resolvents implying the stability of the essential or critical spectra of perturbed C_0-semigroups on Hilbert spaces. We also show how these results apply to transport theory.

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1 Introduction

Let T be the generator of a C_0-semigroup $(U(t))_{t\geq 0}$ on a Banach space X and $K \in \mathcal{L}(X)$, i.e. a linear bounded operator on X. Let $(V(t))_{t\geq 0}$ be the C_0-semigroup generated by $T+K$. It is given by the Dyson-Phillips expansion

$$V(t) = \sum_{j=0}^{\infty} U_j(t), \tag{1.1}$$

where

$$U_0(t) = U(t), \quad U_{j+1}(t) = \int_0^t U_j(t-s)KU_0(s)ds \ (j \geq 0).$$

It is known that

$$\text{the compactness of } U_k(t) \text{ for all } t \geq 0 \tag{1.2}$$

implies the stability of the essential spectrum, i.e. $\sigma_{ess}(V(t)) = \sigma_{ess}(U(t))$, if $k=1$ and the stability of the essential type, i.e. $\omega_{ess}(V(\cdot)) = \omega_{ess}(U(\cdot))$, if $k \geq 1$ (see [17, 24, 29, 30, 31]). On the other hand,

$$\text{the norm continuity of } 0 \leq t \mapsto U_k(t) \tag{1.3}$$

ensures the stability of the critical spectrum, i.e. $\sigma_{crit}(V(t)) = \sigma_{crit}(U(t))$, when $k=1$ and the stability of the critical type, i.e. $\omega_{crit}(V(\cdot)) = \omega_{crit}(U(\cdot))$, when $k \geq 1$ (see [3] for these stability results, and [21] for the properties of the critical spectrum of C_0-semigroups in Banach spaces). Actually, (1.2) and (1.3) are linked by S. Brendle's result:

Theorem 1.1. *[2, Theorem 3.2] Let $\nu > \omega_0(U(\cdot))$ and $k \in \mathbb{N}$. Then the following statements are equivalent.*
- *$U_k(t)$ is compact for all $t \geq 0$.*
- *$0 \leq t \mapsto U_k(t)$ is norm continuous and $R(\nu + i\gamma, T)(KR(\nu + i\gamma, T))^k$ is compact for all $\gamma \in \mathbb{R}$.*

Naturally, an interesting problem is: what conditions, in terms of the resolvents of the generators, imply the norm continuity of $0 \leq t \mapsto U_k(t)$? When the underlying space is a Hilbert space, by the use of a technique of O. El-Mennaoui and K. J. Engel [7], a sufficient condition was given by S. Brendle:

Theorem 1.2. *[2, Theorem 3.1] Assume that X is a Hilbert space. Let $\nu > \omega_0(U(\cdot))$ and $k \in \mathbb{N}$. If*

$$\|R(\nu + i\gamma, T)(KR(\nu + i\gamma, T))^k\| \to 0 \text{ as } |\gamma| \to \infty \tag{1.4}$$

then $0 \leq t \mapsto U_{k+2}(t)$ is norm continuous.

We note that an assumption like (1.4) together with a compactness of some iterate of $R(\lambda, T)K$ allowed D. Song [25], via Gearhart's results [8], to obtain the following estimate for the essential type: $\omega_{ess}(V(\cdot)) \leq \omega_0(U(\cdot))$. That result was also applied by K. Latrach and B. Lods [13] to transport equations. It is clear that we cannot deal with the norm continuity of $0 \leq t \mapsto U_1(t)$ by means of Theorem 1.2. In [35] Q. C. Zhang and F. L. Huang characterize the norm continuity of $0 \leq t \mapsto V(t) - U(t)$ (this is equivalent to the norm continuity of $0 \leq t \mapsto U_1(t)$ [17, Theorem 2.7, p. 18]):

Theorem 1.3. *[35] Assume that X is a separable Hilbert space. Let $\nu > \omega_0(U(\cdot))$. Then $0 \leq t \mapsto V(t) - U(t)$ is norm continuous if and only if*

$$\|R(\nu + i\gamma, T)KR(\nu + i\gamma, T)\| \to 0 \text{ as } |\gamma| \to \infty \tag{1.5}$$

$$\sup_{\|x\| \leq 1} \int_A^\infty \Big[\|R(\nu \pm i\gamma, T)KR(\nu \pm i\gamma, T)x\|^2 + \|R(\nu \pm i\gamma, T^*)K^*R(\nu \pm i\gamma, T^*)x\|^2 \Big] d\gamma \to 0 \text{ as } A \to \infty. \tag{1.6}$$

For a further generalization of this result see [14]. The condition (1.6), of integral type, is not easy to check in practice. However, the authors of [35] mentioned a result due to J. G. Peng *et al* which gives directly the compactness of $V(t) - U(t)$, namely:

Theorem 1.4. *[22] Assume that X is a separable Hilbert space. If*

$$R(\lambda, T)KR(\lambda, T) \text{ is compact for } \lambda \in \rho(T)$$

and

$$\lim_{|\gamma| \to +\infty} \|R(\nu + i\gamma, T)^2K\| + \|R(\nu + i\gamma, T)KR(\nu + i\gamma, T)\| + \|KR(\nu + i\gamma, T)^2\| = 0, \tag{1.7}$$

then $V(t) - U(t)$ is compact for all $t \geq 0$.

We give here sufficient conditions implying the norm continuity of $0 \leq t \mapsto U_1(t)$. More precisely, we show in Lemma 2.1 that if for some $m \in \mathbb{N}^*$ and some $\nu > \omega_0(U(\cdot))$

$$(\mathbf{A_m}) \qquad \sum_{i=0}^{m} \|R(\nu + i\gamma, T)^i K R(\nu + i\gamma, T)^{m-i}\| \to 0 \text{ as } |\gamma| \to \infty,$$

then $0 \leq t \mapsto U_1(t)$ is norm continuous. This result is proved by adapting mathematical techniques used in Brendle's paper [2]. Note that $(\mathbf{A_2})$ corresponds to (1.7). Motivated by problems arising in transport theory [15], we reformulate in Corollary 2.9 condition $(\mathbf{A_1})$ in another way, when T is dissipative, namely:

$$\|K^* R(\nu + i\gamma, T)K\| + \|KR(\nu + i\gamma, T)K^*\| \to 0 \text{ as } |\gamma| \to \infty.$$

The above results enable us to derive some stability results on essential or critical spectra of perturbed semigroups. In the last section, we show how the results obtained in Section 2 apply to transport equations with vacuum boundary conditions (transport equations with boundary operators are dealt with in [15]). We consider the integro-differential equation which governs the distribution of neutrons in a nuclear reactor

$$\frac{\partial \varphi}{\partial t}(x, v, t) + v \cdot \frac{\partial \varphi}{\partial x}(x, v, t) + \sigma(x, v)\varphi(x, v, t) - \int_V k(x, v, v')\varphi(x, v', t)d\mu(v') = 0 \qquad (1.8)$$

with initial and boundary conditions

$$\varphi(x, v, 0) = \varphi_0(x, v), \quad \varphi_{|\Gamma_-}(\cdot, \cdot, t) = 0, \qquad (1.9)$$

where $(x, v) \in \Omega \times V$, Ω is a smooth open subset of $\mathbb{R}^N$ $(N \geq 1)$ and V (the velocity space) is a support of a positive Radon measure μ on $\mathbb{R}^N$. Here

$$\Gamma_- = \{(x, v) \in \partial\Omega \times V;\ v \cdot \eta(x) < 0\},$$

where $\eta(x)$ is the unit outward normal at $x \in \partial\Omega$. The collision frequency $\sigma(\cdot, \cdot) \in L^\infty(\Omega \times V)$ is a non-negative function while $k(\cdot, \cdot, \cdot)$ is the scattering kernel. The unbounded operator, called the streaming operator,

$$T : D(T) \ni \varphi \mapsto -v \cdot \frac{\partial \varphi}{\partial x} - \sigma(x, v)\varphi(x, v)$$

with domain

$$D(T) = \{\varphi \in L^2(\Omega \times V);\ v \cdot \frac{\partial \varphi}{\partial x} \in L^2(\Omega \times V),\ \varphi_{|\Gamma_-} = 0\}$$

is the infinitesimal generator of the so-called streaming C_0-semigroup $(U(t))_{t \geq 0}$ given by

$$U(t)\varphi(x, v) = \begin{cases} e^{-\int_0^t \sigma(x - sv, v)ds}\varphi(x - tv, v) & \text{if } t < \tau(x, v), \\ 0 & \text{otherwise}, \end{cases}$$

where $\tau(x, v) = \inf\{s > 0; (x - sv) \notin \Omega\}$. Moreover, if the collision operator

$$K : L^2(\Omega \times V) \ni \varphi \mapsto \int_V k(x, v, v')\varphi(x, v')d\mu(v') \in L^2(\Omega \times V) \qquad (1.10)$$

is bounded on $L^2(\Omega \times V)$, then $T + K$ generates a C_0-semigroup $(V(t))_{t\geq 0}$ which solves the evolution problem (1.8)-(1.9). Many authors studied the compactness of terms of the Dyson-Phillips expansions when Ω is bounded, we quote the contributions of K. Jörgens [11], I. Vidav [27], G. Greiner [9], J. Voigt [30], P. Takac [26], L. Weis [31], M. Mokhtar-Kharroubi [16][17, Chapter 4] etc... Typically, $U_2(t)$ (or equivalently $R_2(t)$) is compact for a large class of collision operators. It is only recently that the compactness of $U_1(t)$ was proved by M. Mokhtar-Kharroubi [19]. On the other hand, in unbounded domains, due to the lack of compactness, it turned out in [20] that the critical spectrum and the norm continuity of (1.3) are the relevant tools to the spectral analysis of $(V(t))_{t\geq 0}$. It was shown that $0 \leq t \mapsto U_1(t)$ is norm continuous under fairly general assumptions. In this section we show how our functional analytic results provide a new approach to those results and allow also to improve some of them. We note that the essential and critical spectra of $(U(t))_{t\geq 0}$ are known [19, Remark 2] and [20]. Firstly, we single out a general class of measures μ and collision operators K under which the condition $(\mathbf{A_1})$ holds. We prove, for bounded spatial domains, that

$$\sigma_{ess}(V(t)) = \sigma_{ess}(U(t)) \quad (\forall t \geq 0).$$

As announced before, this result was already proved by M. Mokhtar-Kharroubi [19] by investigating the compactness of $U_1(t)$ directly. We also show

$$\sigma_{crit}(V(t)) = \sigma_{crit}(U(t)) \quad (\forall t \geq 0) \tag{1.11}$$

under much more general assumptions than in [20], where (1.11) was established by studying the norm continuity of $0 \leq t \mapsto U_1(t)$ directly. In particular, we answer positively some open problems posed in [20]. Our mathematical analysis relies on density arguments and Fourier analysis. We end up the last section by investigating more general collision operators and show thus how powerful is the resolvent approach in transport theory.

2 Norm continuity of $0 \leq t \mapsto U_1(t)$ in Hilbert spaces

From now on X is a Hilbert space and we denote it by H. Our first result in this section is the following basic lemma:

Lemma 2.1. *Assume there exist $\nu > \omega_0(U(\cdot))$ and $m \in \mathbb{N}^*$ such that*

$(\mathbf{A_m})$ $\quad \|R(\nu + i\gamma, T)^i K R(\nu + i\gamma, T)^{m-i}\| \to 0$ *as* $|\gamma| \to \infty$ *for* $i = 0, 1, \cdots, m.$

Then $0 \leq t \mapsto U_1(t)$ is norm continuous.

A first consequence of this lemma is the following stability result of the critical spectrum.

Theorem 2.2. *Assume there exist $\nu > \omega_0(U(\cdot))$ and $m \in \mathbb{N}^*$ such that*

$$\|R(\nu + i\gamma, T)^i K R(\nu + i\gamma, T)^{m-i}\| \to 0 \text{ as } |\gamma| \to \infty \text{ for } i = 0, 1, \cdots, m.$$

Then $\sigma_{crit}(V(t)) = \sigma_{crit}(U(t))$ for $t \geq 0$.

Proof. According to [3, Theorem 4.5], if $0 \leq t \mapsto V(t) - U(t)$ is norm continuous, then the two semigroups have the same critical spectrum. On the other hand we know, by [17, Theorem 2.7, p 18], that $0 \leq t \mapsto V(t) - U(t)$ is norm continuous if and only if $0 \leq t \mapsto U_1(t)$ is. Then the theorem follows from Lemma 2.1. ■

A second consequence of Lemma 2.1 is the following stability result of the essential spectrum.

Theorem 2.3. *Assume there exist $\nu > \omega_0(U(\cdot))$ and $m \in \mathbb{N}^*$ such that*

- $\|R(\nu + i\gamma, T)^i K R(\nu + i\gamma, T)^{m-i}\| \to 0$ *as* $|\gamma| \to \infty$ *for* $i = 0, 1, \cdots, m$.
- $R(\nu + i\gamma, T) K R(\nu + i\gamma, T)$ *is compact for all* $\gamma \in \mathbb{R}$.

Then $\sigma_{ess}(V(t)) = \sigma_{ess}(U(t))$ *for all* $t \geq 0$.

Proof. It is well known that two bounded operators whose difference is compact have the same essential spectrum. Actually, by [17, Theorem 2.6, p. 16], the difference $V(t) - U(t)$ is compact for all $t \geq 0$ if and only if $U_1(t)$ is, which, by Theorem 1.1, is equivalent to $0 \leq t \mapsto U_1(t)$ being norm continuous and $R(\nu + \gamma, T) K R(\nu + \gamma, T)$ being compact for all $\gamma \in \mathbb{R}$ ($\nu > \omega_0(U(\cdot))$). Summarizing all that, the proof is an immediate consequence of Lemma 2.1. ■

Our proof of lemma 2.1 is inspired and adapted from [2]. Before giving it, we recall some facts. We start with

Lemma 2.4. *[28] For all $t, s \geq 0$ and $p \in \mathbb{N}$ we have the following identity*

$$U_p(t + s) = \sum_{j=0}^{p} U_j(t) U_{p-j}(s).$$

Using this lemma we can define a new C_0-semigroup on the Hilbert space $\mathcal{H} := H \times H$. Let

$$\mathcal{U}(t) := \begin{pmatrix} U_0(t) & U_1(t) \\ 0 & U_0(t) \end{pmatrix}$$

for all $t \geq 0$, then $(\mathcal{U}(t))_{t \geq 0}$ is a C_0-semigroup with generator

$$\mathcal{T} = \begin{pmatrix} T & K \\ 0 & T \end{pmatrix}$$

and domain $D(\mathcal{T}) = D(T) \times D(T)$. The resolvent of $\mathcal{T}$ is given by

$$R(\lambda, \mathcal{T}) = \begin{pmatrix} R(\lambda, T) & R(\lambda, T) K R(\lambda, T) \\ 0 & R(\lambda, T) \end{pmatrix}$$

for $Re\lambda > \omega_0(U(\cdot))$ (see [2]).
The following lemma provides a representation of $U_1(t)$ in terms of the resolvent. It contains an essential piece of analysis for the proof of Lemma 2.1.

Lemma 2.5. *Let $\nu > \omega_0(U(\cdot))$ and $n \in \mathbb{N}^*$. For all $x \in H$,*

$$U_1(t)x = \frac{n!}{2t^n \pi} \lim_{A \to \infty} \int_{-A}^{A} e^{(\nu + i\gamma)t} \sum_{i=0}^{n} R(\nu + i\gamma, T)^{i+1} K R(\nu + i\gamma, T)^{n-i+1} x d\gamma,$$

where the integrals converge uniformly for $t \in \mathbb{R}$.

Proof. Let $n \in \mathbb{N}^*$. By [33, Theorem 1.1], for all $\kappa \in \mathcal{H}$

$$\mathcal{U}(t)\kappa = \frac{n!}{2t^n\pi} \lim_{A\to\infty} \int_{-A}^{A} e^{(\nu+i\gamma)t} R(\nu+i\gamma, \mathcal{T})^{n+1}\kappa d\gamma,$$

where the integrals converge uniformly for $t \in \mathbb{R}$. By considering the components of $\mathcal{U}(t)\kappa$ and $R(\nu+i\gamma, \mathcal{T})^{n+1}\kappa$ the result follows. ■

Now we are ready to give the:

Proof of Lemma 2.1. Let $\nu > \omega_0(U(\cdot))$. Define the operator $U_1^A(t)$ on H as follows

$$U_1^A(t)x = \frac{m!}{2t^m\pi} \int_{-A}^{A} e^{(\nu+i\gamma)t} \sum_{i=0}^{m} R(\nu+i\gamma, T)^{i+1} K R(\nu+i\gamma, T)^{m-i+1} x d\gamma.$$

We are going to show that

$$\|U_1(t) - U_1^A(t)\| \to 0 \text{ as } A \to +\infty \tag{2.1}$$

uniformly for $t \in [\delta, \frac{1}{\delta}]$ $(\delta > 0)$ and that, for each $A > 0$,

$$0 \le t \mapsto U_1^A(t) \in \mathcal{L}(H) \tag{2.2}$$

is norm continuous. Let us show (2.1). According to Lemma 2.5 we have

$$(U_1(t) - U_1^A(t))x = \frac{m!}{2t^m\pi} \int_{|\gamma|>A} e^{(\nu+i\gamma)t} \sum_{i=0}^{m} R(\nu+i\gamma, T)^{i+1} K R(\nu+i\gamma, T)^{m-i+1} x d\gamma.$$

Let x, $x^* \in H$, using Cauchy-Schwartz's inequality and Plancherel's theorem [10, Lemma 2], we obtain

$$\begin{aligned}
&\frac{2t^m\pi}{m!} |\langle (U_1(t) - U_1^A(t))x, x^*\rangle| \\
\le & \int_{|\gamma|>A} \sum_{i=0}^{m} |\langle R(\nu+i\gamma, T)^{i+1} K R(\nu+i\gamma, T)^{m-i+1} x, x*\rangle| d\gamma \\
= & \int_{|\gamma|>A} \sum_{i=0}^{m} |\langle R(\nu+i\gamma, T)^{i} K R(\nu+i\gamma, T)^{m-i} R(\nu+i\gamma, T)x, R(\nu-i\gamma, T^*)x*\rangle| d\gamma \\
\le & \sum_{i=0}^{m} \sup_{|\gamma|>A} \|R(\nu+i\gamma, T)^{i} K R(\nu+i\gamma, T)^{m-i}\| \\
& \qquad \times \Big[\int_{-\infty}^{+\infty} \|R(\nu+i\gamma, T)x\|^2 d\gamma\Big]^{\frac{1}{2}} \Big[\int_{-\infty}^{+\infty} \|R(\nu-i\gamma, T^*)x^*\|^2 d\gamma\Big]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{m} \sup_{|\gamma|>A} \|R(\nu+i\gamma,T)^i K R(\nu+i\gamma,T)^{m-i}\| \\
&\qquad \times \Big[\sqrt{2\pi}\int_0^{+\infty} e^{-2\nu s}\|U(s)x\|^2 ds\Big]^{\frac{1}{2}} \Big[\sqrt{2\pi}\int_0^{+\infty} e^{-2\nu s}\|U^*(s)x^*\|^2 ds\Big]^{\frac{1}{2}} \\
&\le 2\pi C \sum_{i=0}^{m} \sup_{|\gamma|>A} \|R(\nu+i\gamma,T)^i K R(\nu+i\gamma,T)^{m-i}\| \|x\| \|x^*\|,
\end{aligned}
$$

where C is some positive constant. Thus

$$
\|U_1(t) - U_1^A(t)\| \le \frac{Cm!}{t^m} \sum_{i=0}^{m} \sup_{|\gamma|>A} \|R(\nu+i\gamma,T)^i K R(\nu+i\gamma,T)^{m-i}\|
$$

which, by assumptions, converges to zero as A goes to infinity uniformly for $t \in [\delta, \frac{1}{\delta}]$. It remains to prove (2.2). To this end, let $t > 0$. We have

$$
\begin{aligned}
& \Big\| \frac{2\pi(t+h)^m}{m!} U_1^A(t+h) - \frac{2\pi t^m}{m!} U_1^A(t) \Big\| \\
\le & \sum_{i=0}^{m} \sup_{|\gamma|\le A} \|R(\nu+i\gamma,T)^{i+1} K R(\nu+i\gamma,T)^{m-i+1}\| \int_{-A}^{A} |e^{(\nu+i\gamma)(t+h)} - e^{(\nu+i\gamma)t}| d\gamma.
\end{aligned}
$$

One sees that the last term tends to zero as h goes to zero. The proof is now achieved. ■

Remark 2.6. Lemma 2.1 is in the spirit of the result proved by P.You [34], which characterizes the norm continuous semigroups in Hilbert spaces in terms of the resolvent of their generators. The technique we use in the proof is due to O. El-Mennaoui and K. J. Engel [7] (see also [1, Theorem 3.13.2, p. 205]).

Remark 2.7. For $m = 2$, Theorem 2.3 coincides with Theorem 1.4.

Under the dissipativity of T, we can reformulate $(\mathbf{A_1})$ in more workable form in some cases (see Section 3.2 and [15]). Let $\nu > 0$. The following lemma shows that, under the dissipativity of T, one can control the asymptotic behaviour of $\|R(\nu+i\gamma,T)K\|$ in the axis $Re\lambda = \nu$ by that of $\|K^*R(\nu+i\gamma,T)K\|$, and accordingly the asymptotic behaviour of $\|KR(\nu+i\gamma,T)\|$ by that of $\|KR(\nu+i\gamma,T)K^*\|$.

Lemma 2.8. *Assume that T is dissipative. Then for $\nu > 0$*

$$
\sqrt{\nu}\|R(\nu+i\gamma,T)K\| \le \sqrt{\|K^*R(\nu+i\gamma,T)K\|} \tag{2.3}
$$

and

$$
\sqrt{\nu}\|KR(\nu+i\gamma,T)\| \le \sqrt{\|KR(\nu+i\gamma,T)K^*\|}. \tag{2.4}
$$

Proof. Since T is dissipative,

$$
\begin{aligned}
|\langle(\lambda - T)y, y\rangle| &\ge Re\langle(\lambda - T)y, y\rangle \\
&\ge Re\lambda\|y\|^2
\end{aligned}
$$

for all $y \in D(T)$, where $\langle \cdot, \cdot \rangle$ stands for the scalar product in H. In particular, for $\lambda = \nu + i\gamma$ and $y = R(\nu + i\gamma, T)Kx$ with $x \in H$

$$\begin{aligned}\nu \|R(\nu + i\gamma, T)Kx\|^2 &\leq |\langle Kx, R(\nu + i\gamma, T)Kx\rangle| \\ &= |\langle x, K^* R(\nu + i\gamma, T)Kx\rangle| \\ &\leq \|K^* R(\nu + i\gamma, T)K\| \|x\|^2\end{aligned}$$

which shows (2.3). Inequality (2.4) follows by duality. Indeed, since T^* is dissipative, by (2.3) we obtain

$$\begin{aligned}\sqrt{\nu}\|KR(\nu + i\gamma, T)\| &= \sqrt{\nu}\|R(\nu - i\gamma, T^*)K^*\| \\ &\leq \sqrt{\|KR(\nu - i\gamma, T^*)K^*\|} \\ &= \sqrt{\|KR(\nu + i\gamma, T)K^*\|},\end{aligned}$$

which ends the proof. ■

Taking into account of (2.3) and (2.4), it follows from Lemma 2.1:

Corollary 2.9. *Assume that T is dissipative and that for some $\nu > 0$*

$$\|K^* R(\nu + i\gamma, T)K\| \to 0 \quad \textit{and} \quad \|KR(\nu + i\gamma, T)K^*\| \to 0 \quad \textit{as } |\gamma| \to \infty,$$

then $0 \leq t \mapsto U_1(t)$ is norm continuous.

Remark 2.10. Actually, Lemma 2.8 remains true if we assume that T is sum of a dissipative operator T_1 and a bounded operator B. In this case we can prove that for $\nu > \|B\|$, $\sqrt{(\nu - \|B\|)}\|R(\nu + i\gamma, T)K\| \leq \sqrt{\|K^* R(\nu + i\gamma, T)K\|}$ and $\sqrt{(\nu - \|B\|)}\|KR(\nu + i\gamma, T)\| \leq \sqrt{\|KR(\nu + i\gamma, T)K^*\|}$. Indeed, Since T_1 is dissipative,

$$\begin{aligned}|\langle (\lambda - T_1 - B)y, y\rangle| &\geq Re\langle (\lambda - T_1 - B)y, y\rangle \\ &\geq Re\lambda \|y\|^2 - Re\langle By, y\rangle \\ &\geq (Re\lambda - \|B\|)\|y\|^2\end{aligned}$$

for all $y \in D(T)$. By taking $\lambda = \nu + i\gamma$ with $\nu > \|B\|$ and $y = R(\nu + i\gamma, T)Kx$ with $x \in H$ we get

$$\sqrt{(\nu - \|B\|)}\|R(\nu + i\gamma, T)Kx\| \leq \sqrt{\|K^* R(\nu + i\gamma, T)K\|}\|x\|.$$

The second inequality follows by duality. Consequently, if $T = T_1 + M$ with T_1 is dissipative operator and B is bounded and if for some $\nu > \|B\|$,

$$\|K^* R(\nu + i\gamma, T)K\| \to 0 \quad \text{and} \quad \|KR(\nu + i\gamma, T)K^*\| \to 0 \quad \text{as } |\gamma| \to \infty,$$

then $0 \leq t \mapsto U_1(t)$ is norm continuous.

3 Application to transport theory

In this section we study the stability of the essential or the critical spectrum of the streaming semigroup when its generator is perturbed by the collision operator. In Theorem 3.6, we find again by a resolvent approach, an optimal result [19, Theorem 1 and Corollary1] obtained recently by M. Mokhtar-Kharroubi by means of semigroup approach. We also answer positively some open problems posed in [20] (see Remarks 3.9 and 3.11).
Let $1 < p < \infty$. We consider the collision operator K as a mapping

$$K(\cdot) : \Omega \ni x \mapsto K(x) \in \mathcal{L}(L^p(V, \mu)),$$

where

$$K(x) : L^p(V) \ni \varphi \mapsto \int_V k(x, v, v')\varphi(v')d\mu(v') \in L^p(V).$$

We assume that $K(\cdot)$ is strongly measurable, i.e.

$$\Omega \ni x \mapsto K(x)\psi \in L^p(V, \mu) \text{ is measurable for any } \psi \in L^p(V, \mu)$$

and

$$\Omega \ni x \mapsto \|K(x)\|_{\mathcal{L}(L^p(V))} \text{ is essentially bounded.}$$

Then, we define a collision operator by

$$K : L^p(\Omega \times V) \ni \varphi \mapsto K(x)\varphi(x),$$

where we make the identification $L^p(\Omega \times V) := L^p(\Omega; L^p(V))$. It follows from [19] that $K \in \mathcal{L}(L^p(\Omega \times V))$ and $\|K\|_{\mathcal{L}(L^p(\Omega\times V))} = \operatorname*{ess\,sup}_{x\in\Omega} \|K(x)\|_{\mathcal{L}(L^p(V))}$. In what follows we will use the concept of regular operator:

Definition 3.1. *[19] Let $1 < p < \infty$. A collision operator K is said to be regular if:*

(i) $\{K(x);\ x \in \Omega\}$ is a collectively compact set of operators on $L^p(V)$, i.e.

$$\{K(x)\psi; \quad x \in \Omega, \|\psi\|_{L^p(V)} \le 1\}$$

is relatively compact in $L^p(V)$.

(ii) For every $\psi' \in L^{p'}(V)$, $\{K'(x)\psi';\ x \in \Omega\}$ is relatively compact in $L^{p'}(V)$.

We recall:

Lemma 3.2. *[19] The class of regular collision operators is the closure in $\mathcal{L}(L^p(\Omega \times V))$ of the subclass of collision operators with kernels of the form*

$$k(x, v, v') = \sum_{i\in I} \alpha_i(x) f_i(v) g_i(v'), \tag{3.1}$$

with $f_i \in L^p(V)$, $g_i \in L^{p'}(V)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) and $\alpha_i \in L^\infty(\Omega)$ (I finite).

3.1 Technical results

In order to apply the above stability results, our main task is to find sufficient conditions under which $(\mathbf{A_1})$ holds. This will be provided by the following basic lemma:

Lemma 3.3. *Let $p = 2$ and let Ω be convex. We assume that the collision operator is regular, that the affine hyperplanes (i.e. the translated hyperplanes) have zero μ-measure and that $\sigma(x, v)$ is space homogeneous (i.e. $\sigma(x, v) = \sigma(v)$). Then for all $\nu > \omega_0(U(\cdot))$*

$$\|KR(\nu + i\gamma, T)\| \to 0 \text{ and } \|R(\nu + i\gamma, T)K\| \to 0 \text{ as } |\gamma| \to \infty.$$

Before giving the proof, we recall:

Lemma 3.4. *[17, Lemma 3.3, p. 39] Let μ be a finite Radon on $\mathbb{R}^N$ such that the affine hyperplanes have zero μ-measure. Then*

$$\lim_{\varepsilon\to 0} \sup_{(e,e_1)\in S^N} \mu\{v \in \mathbb{R}^N; |v \cdot e + e_1| \leq \varepsilon\} = 0,$$

where $e \in \mathbb{R}^N$, $e_1 \in \mathbb{R}$ and S^N is the unit sphere of $\mathbb{R}^{N+1}$.

Proof of Lemma 3.3. It follows from the Resolvent identity

$$R(\varsigma + i\gamma, T) - R(\nu + i\gamma, T) = (\nu - \varsigma)R(\varsigma + i\gamma, T)R(\nu + i\gamma, T) \quad \text{for } \varsigma, \nu > \omega_0(U(\cdot)),$$

that if $\lim_{|\gamma|\to\infty}(\|R(\nu + i\gamma, T)K\| + \|KR(\nu + i\gamma, T)\|) = 0$ holds for some $\nu > \omega_0(U(\cdot))$, it remains true for all $\varsigma > \omega_0(U(\cdot))$. Thus it is no restriction to assume that $\nu > 0$ (with this assumption the operator $R(\nu + i\gamma)$ bellow will be well defined). We note that the operators $R(\nu + i\gamma, T)K$ and $KR(\nu + i\gamma, T)$ depend continuously on $K \in \mathcal{L}(L^2(\Omega \times V))$, uniformly for $\gamma \in \mathbb{R}$. Then, using Lemma 3.2, we may assume without loss of generality that the collision operator K has a kernel of degenerate form, i.e. $k(x, v, v') = \sum_{i\in I} \alpha_i(x)f_i(v)g_i(v')$ with $f_i(\cdot)$, $g_i(\cdot) \in L^2(V)$ and $\alpha_i(\cdot) \in L^\infty(\Omega)$ (I finite). By density again, we may suppose that $f_i(\cdot)$, $g_i(\cdot)$ are continuous with compact supports. We can (by linearity) also suppose that the last sum contains one term, say $\alpha(x)f(v)g(v')$. Let us show first

$$\|R(\nu + i\gamma, T)K\| \to 0 \text{ as } |\gamma| \to \infty. \tag{3.2}$$

We recall that $R(\lambda, T)\varphi(x, v) = \int_0^{\tau(x,v)} e^{-(\lambda+\sigma(v))t}\varphi(x - tv, v)dt$ for $\varphi \in L^2(\Omega \times V)$, where $\tau(x, v) = \inf\{s > 0;\ x - sv \notin \Omega\}$. Hence the convexity of Ω allows to write

$$R(\nu + i\gamma, T)K = \mathcal{R}R(\nu + i\gamma, T_\infty)\mathcal{E}K,$$

where $\mathcal{E} : L^2(\Omega \times V) \to L^2(\mathbb{R}^N \times V)$ is the trivial extension (by zero) to $\mathbb{R}^N \times V$, $\mathcal{R} : L^2(\mathbb{R}^N \times V) \to L^2(\Omega \times V)$ is the restriction operator and T_∞ is the streaming operator on the whole space $\mathbb{R}^N \times V$ with collision frequency $\sigma(v)$. Define K_∞ as follows:

$$K_\infty : L^2(\mathbb{R}^N \times V) \ni \varphi \mapsto \int_V \chi_\Omega(x)\alpha(x)f(v)g(v')\varphi(x, v')d\mu(v') \in L^2(\mathbb{R}^N \times V).$$

Then

$$\mathcal{R}R(\nu + i\gamma, T_\infty)\mathcal{E}K = \mathcal{R}R(\nu + i\gamma, T_\infty)K_\infty\mathcal{E}.$$

The operator $R(\nu+i\gamma,T_\infty)K_\infty$ splits as

$$R(\nu+i\gamma,T_\infty)K_\infty = M_2R(\nu+i\gamma)M_1,$$

where

$$M_1 : L^2(\mathbb{R}^N\times V)\ni\varphi\mapsto\alpha(x)\chi_\Omega(x)\int\limits_V\varphi(x,v')g(v')d\mu(v')\in L^2(\mathbb{R}^N),$$

$$M_2 : L^2(\mathbb{R}^N\times\widetilde{V})\ni\varphi\mapsto f(v)\varphi(x,v)\in L^2(\mathbb{R}^N\times V)$$

(the function φ is extended by zero outside of its domain) and

$$R(\nu+i\gamma) : L^2(\mathbb{R}^N)\ni\varphi\mapsto\int_0^{+\infty}e^{-t(\nu+i\gamma+\sigma(v))}\varphi(x-tv)dt\in L^2(\mathbb{R}^N\times\widetilde{V})$$

where $\widetilde{V}$ is the support of f. Hence, it suffices to deal with $R(\nu+i\gamma)$. On the other hand, for $\varphi\in L^2(\mathbb{R}^N)$, $\psi\in L^2(\mathbb{R}^N\times\widetilde{V})$, the equation

$$\psi = R(\nu+i\gamma)\varphi$$

is equivalent to

$$v\cdot\frac{\partial\psi}{\partial x}+\sigma(v)\psi+(\nu+i\gamma)\psi=\varphi$$

or equivalently

$$\widehat{\psi}=\frac{1}{(\sigma(v)+\nu)+i(\xi\cdot v+\gamma)}\widehat{\varphi},$$

where $\widehat{\varphi}$ and $\widehat{\psi}$ are respectively the L^2-Fourier transform of φ and ψ with respect to the space variable. Let

$$\widehat{R}(\nu+i\gamma) : L^2(\mathbb{R}^N)\ni\phi(\xi)\mapsto\frac{\phi(\xi)}{(\sigma(v)+\nu)+i(\xi\cdot v+\gamma)}\in L^2(\mathbb{R}^N\times\widetilde{V}).$$

Thanks to Parseval's equality, it is sufficient to show that

$$\lim_{|\gamma|\to\infty}\|\widehat{R}(\nu+i\gamma)\|_{\mathcal{L}(L^2(\mathbb{R}^N),L^2(\mathbb{R}^N\times\widetilde{V}))}=0.$$

We have

$$\begin{aligned}\|\widehat{R}(\nu+i\gamma)\phi\|^2 &= \int\limits_{\mathbb{R}^N}\int\limits_{\widetilde{V}}\frac{|\phi(\xi)|^2}{(\sigma(v)+\nu)^2+(\xi\cdot v+\gamma)^2}d\mu(v)d\xi\\ &\leq \sup_{\xi\in\mathbb{R}^N}\int\limits_{\widetilde{V}}\frac{1}{(\sigma(v)+\nu)^2+(\xi\cdot v+\gamma)^2}d\mu(v)\|\phi\|^2.\end{aligned}$$

Now, our goal is to show

$$\lim_{|\gamma|\to\infty}\int\limits_{\widetilde{V}}\frac{1}{(\sigma(v)+\nu)^2+(\xi\cdot v+\gamma)^2}d\mu(v)=0$$

uniformly for $\xi \in \mathbb{R}^N$. Let $\varepsilon > 0$ arbitrary. Introduce polar coordinates, in $\mathbb{R}^{N+1}$, $(\xi, \gamma) = |(\xi, \gamma)|(e, e_1)$, $e = \xi/\sqrt{|\xi|^2 + \gamma^2}$ and $e_1 = \gamma/\sqrt{|\xi|^2 + \gamma^2}$, which give

$$
\begin{aligned}
&\int_{\widetilde{V}} \frac{1}{(\sigma(v)+\nu)^2 + (\xi \cdot v + \gamma)^2} d\mu(v) \\
= &\int_{\widetilde{V} \cap \{|e \cdot v + e_1| \leq \varepsilon\}} \frac{1}{(\sigma(v)+\nu)^2 + (|\xi|^2 + |\gamma|^2)(e \cdot v + e_1)^2} d\mu(v) \\
&\quad + \int_{\widetilde{V} \cap \{|e \cdot v + e_1| > \varepsilon\}} \frac{1}{(\sigma(v)+\nu)^2 + (|\xi|^2 + |\gamma|^2)(e \cdot v + e_1)^2} d\mu(v) \\
\leq &\sup_{(e,e_1) \in S^N} \int_{\widetilde{V} \cap \{|e \cdot v + e_1| \leq \varepsilon\}} \frac{1}{(\sigma(v)+\nu)^2} d\mu(v) \\
&\quad + \int_{\widetilde{V} \cap \{|e \cdot v + e_1| > \varepsilon\}} \frac{1}{(\sigma(v)+\nu)^2 + (|\xi|^2 + |\gamma|^2)\varepsilon^2} d\mu(v) \\
=: &\; J_1 + J_2.
\end{aligned}
$$

According to Lemma 3.4, J_1 is arbitrarily small for ε small enough. We choose ε small enough and consider the second term J_2. The last is majorized by

$$
\frac{\mu\{\widetilde{V}\}}{(|\xi|^2 + |\gamma|^2)\varepsilon^2}
$$

which goes to zero as $|\gamma| \to \infty$ uniformly for $\xi \in \mathbb{R}^N$. This ends the proof of (3.2). Now, we have

$$
\|KR(\nu + i\gamma, T)\| = \|R(\nu - i\gamma, T^*)K^*\|, \tag{3.3}
$$

where T^* is the adjoint of T and is given by :

$$
T^* : D(T^*) \ni \varphi \mapsto v \cdot \frac{\partial \varphi}{\partial x} - \sigma(v)\varphi \in L^2(\Omega \times V)
$$

with $D(T^*) = \{\varphi \in L^2(\Omega \times V); v \cdot \frac{\partial \varphi}{\partial x} \in L^2(\Omega \times V),\ \varphi_{|\Gamma_+} = 0\}$, where $\Gamma_+ = \{(x, v) \in \partial\Omega \times V;\ v \cdot \eta(x) > 0\}$. Proceeding as above, we can prove that

$$
\|R(\nu - i\gamma, T^*)K^*\| \to 0 \text{ as } |\gamma| \to \infty
$$

which ends the proof in virtue of (3.3). ■

The following proposition gives some necessary conditions, on the velocity measure μ, for the validity of Lemma 3.3.

Proposition 3.5. *Let μ be a finite Radon measure and let $\Omega = \mathbb{R}^N$. We assume that $\sigma = 0$ and that*

$$
K : L^2(\mathbb{R}^N \times V) \ni \varphi \mapsto \int_V \varphi(x, v')d\mu(v').
$$

If there exists a hyperplane $H = \{v \in \mathbb{R}^N; v \cdot \widetilde{e} = \widetilde{a}\}$ with positive μ-measure, where $\widetilde{e} \in S^{N-1}$ and $\widetilde{a} \in (0, +\infty)$, then

$$
\overline{\lim_{|\gamma| \to \infty}} \|R(\nu + i\gamma, T)K\| > 0
$$

for all $\nu > \omega_0(U(\cdot))$.

Proof. As in the proof of Lemma 3.3 by the Resolvent identity, it is enough to prove the lemma for $\nu = 1$. By the Fourier transform (see the proof of Lemma 3.3) this amounts to showing that

$$\overline{\lim_{|\gamma|\to\infty}} \|\widehat{R}(1+i\gamma)\| > 0. \tag{3.4}$$

Indeed, if (3.4) is true, then there exist a normalized sequence $(\phi_n)_n \subset L^2(\mathbb{R}^N)$ and $(\gamma_n)_n \subset \mathbb{R}$ tending to $+\infty$ or $-\infty$ such that

$$\sup_{n\in\mathbb{N}} \|\widehat{R}(1+i\gamma_n)\phi_n\| > 0.$$

By Parseval's equality there exists a normalized sequence $(\varphi_n)_n \subset L^2(\mathbb{R}^N)$ such that

$$\sup_{n\in\mathbb{N}} \|R(1+i\gamma_n)\varphi_n\| > 0.$$

Set $\widetilde{\varphi}_n(x,v) = \varphi_n(x)/\sqrt{\mu(V)}$ (with the notations of the proof of Lemma 3.3, $\widetilde{V} = V$). Then

$$\|\widetilde{\varphi}_n\|_{L^2(\mathbb{R}^N\times V)} = 1 \text{ and } R(1+i\gamma_n,T)K\widetilde{\varphi}_n = \sqrt{\mu(V)}R(1+i\gamma_n)\varphi_n.$$

Hence

$$\overline{\lim_{|\gamma|\to\infty}} \|R(1+i\gamma,T)K\| \geq \sup_{n\in\mathbb{N}} \|R(1+i\gamma_n,T)K\widetilde{\varphi}_n\| = \sqrt{\mu(V)}\sup_{n\in\mathbb{N}} \|R(1+i\gamma_n)\varphi_n\| > 0.$$

Let us show (3.4). For all $n \in \mathbb{N}^*$ define the normalized function

$$\phi_n(\xi) = \frac{1}{|B(0,1)|^{\frac{1}{2}}}\chi_{B(n\widetilde{e},1)}(\xi) \in L^2(\mathbb{R}^N),$$

where $|B(0,1)|$ is the volume of unit ball in $\mathbb{R}^N$ and $B(n\widetilde{e},1)$ is the ball centered at $n\widetilde{e}$ with radius 1. Then

$$\begin{aligned}
\|\widehat{R}(1-in\widetilde{a})\phi_n\|^2 &= \int_{\mathbb{R}^N}\int_V \frac{|\phi_n(\xi)|^2}{1+(\xi\cdot v - n\widetilde{a})^2}d\mu(v)d\xi \\
&\geq \int_{\mathbb{R}^N}\int_H \frac{|\phi_n(\xi)|^2}{1+(\xi\cdot v - n\widetilde{e}\cdot v)^2}d\mu(v)d\xi \\
&= \int_{\mathbb{R}^N}\int_H \frac{|\phi_n(\xi)|^2}{1+((\xi - n\widetilde{e})\cdot v)^2}d\mu(v)d\xi \\
&\geq \int_{\mathbb{R}^N}\int_H \frac{|\phi_n(\xi)|^2}{1+|v|^2}d\mu(v)d\xi \\
&= \int_H \frac{1}{1+|v|^2}d\mu(v)
\end{aligned}$$

so

$$\|\widehat{R}(1-in\widetilde{a})\|^2 \geq \int_H \frac{1}{1+|v|^2}d\mu(v),$$

which finishes the proof. ∎

3.2 Stability of the essential spectrum

Our main result in this subsection is:

Theorem 3.6. *Let $p = 2$ and let Ω be bounded (not necessarily convex). We assume that μ is such that the affine hyperplanes have zero μ-measure and that the collision operator is regular. Then $V(t) - U(t)$ is compact for all $t \geq 0$ and consequently $\sigma_{ess}(V(t)) = \sigma_{ess}(U(t))$ for all $t \geq 0$.*

Proof. We have to show that $V(t) - U(t)$ is compact for all $t \geq 0$ or equivalently, by [17, Theorem 2.6, p. 16], that $U_1(t)$ is compact for all $t \geq 0$. $U_1(t)$ depends (linearly and) continuously, in the operator norm topology, on the collision operator K. Then, using Lemma 3.2 and linearity we may assume that

$$K : L^2(\Omega \times V) \ni \varphi \mapsto \alpha(x) \int_V f(v)g(v')\varphi(x, v')d\mu(v'),$$

where $f, g \in L^2(V)$, $\alpha \in L^\infty(\Omega)$, and f, g and α are non-negative. Let $d > 0$ be such that Ω is included in the ball $B(0, d) := \{x;\ |x| < d\}$. Let $\widetilde{U_1(t)}$ stand for the operator $U_1(t)$ associated to $\Omega = B(0, d)$, the collision frequency $\widetilde{\sigma} = 0$ and the scattering kernel $\|\alpha\|_\infty f(v)g(v')$. Then one can see that

$$U_1(t) \leq \widetilde{\mathcal{R}}\widetilde{U_1(t)}\widetilde{\mathcal{E}}$$

in the lattice sense, where $\widetilde{\mathcal{E}} : L^2(\Omega \times V) \to L^2(B(0, d) \times V)$ is the trivial extension operator while $\widetilde{\mathcal{R}} : L^2(B(0, d) \times V) \to L^2(\Omega \times V)$ is the restriction operator. So according to [6] the compactness of $\widetilde{U_1(t)}$ for all $t \geq 0$ implies that of $U_1(t)$. Therefore, it is no restriction to assume that

$$\Omega \text{ is bounded and convex, and } \sigma \text{ is space homogeneous.}$$

In this case Lemma 3.3 with Lemma 2.1 guarantee us that $0 \leq t \to U_1(t)$ is norm continuous. On the other hand, we know from [17, Theorem 4.1, p. 57] that $KR(\nu + i\gamma, T)$ is compact for all $\gamma \in \mathbb{R}$. Hence from Theorem 1.1, $U_1(t)$ is compact for all $t \geq 0$. This ends the proof. ∎

Remark 3.7. For bounded Ω, the essential spectrum of $U(t)$ is provided in [19, Remark 2]:

$$\sigma_{ess}(U(t)) = \{\mu \in \mathbb{C};\ \ |\mu| \leq e^{-\lambda^* t}\},$$

where $\lambda^* = \lim_{t\to\infty} \inf_{\{(x,v);\ t<\tau(x,v)\}} \frac{1}{t}\int_0^t \sigma(x - sv, v)ds$, with $\tau(x, v) = \inf\{s > 0;\ \ x - sv \notin \Omega\}$.

3.3 Stability of the critical spectrum

In this subsection we deal with the stability of the critical spectrum of $U(t)$. We note that the latter is described in [20, Theorem 7]. First, we consider

Convex spatial domains

Here we assume that the collision frequency is homogeneous, the non homogeneous case will be treated hereafter.

Theorem 3.8. *Let $p = 2$ and let Ω be convex (not necessarily bounded). We assume μ is such that the affine hyperplanes have zero μ-measure, the collision operator is regular and the collision frequency σ is space homogeneous. Then the mapping $0 \leq t \mapsto V(t) - U(t)$ is norm continuous and consequently $\sigma_{crit}(V(t)) = \sigma_{crit}(U(t))$ for all $t \geq 0$.*

Proof. The proof follows from Theorem 2.2 and Lemma 3.3. ■

Remark 3.9. Theorem 3.8 generalizes [20, Theorem 8] and answers positively Problem 2 in [20].

Arbitrary spatial domains

We have used a domination argument to prove the compactness of $V(t) - U(t)$ when Ω is not convex or the collision frequency σ is not homogeneous. But when we deal with the norm continuity of $0 \leq t \mapsto V(t) - U(t)$ it is clear that we cannot invoke a domination argument. In the following, we take the advantage of the dissipativity of the streaming operator T in order to remove the convexity assumption on Ω and the homogeneity assumption on σ, in the particular case where $d\mu(v) = dv$ is the Lebesgue measure on $\mathbb{R}^N$.

Theorem 3.10. *Let $p = 2$ and let Ω be a (not necessarily convex) subset of $\mathbb{R}^N$. We assume that the collision operator is regular, μ is the Lebesgue measure and the collision frequency can be approximated in $L^\infty(\Omega \times V)$ by degenerate collision frequencies of the form $\sum_{i\in I} \sigma_1^i(x)\sigma_2^i(v)$ (I finite). Then $\sigma_{crit}(V(t)) = \sigma_{crit}(U(t))$ for all $t \geq 0$.*

Proof. Since T is dissipative, from Lemma 2.8 and Lemma 3.13 below we get that

$$\|KR(\nu + i\gamma, T)\| + \|R(\nu + i\gamma, T)K\| \to 0 \text{ as } |\gamma| \to \infty, \text{ for all } \nu > 0.$$

This implies, by using Theorem 2.2, that $\sigma_{crit}(V(t)) = \sigma_{crit}(U(t))$ for all $t \geq 0$. ■

Remark 3.11. Theorem 3.10 answers, at least for the Lebesgue measure, Problems 1 and 3 in [20] about the relevance of the convexity assumption on Ω and the homogeneity assumption on σ.

Remark 3.12. For a general (but abstract) criterion to approximate a collision frequency by a degenerate collision frequencies see [18, Theorem A.4]. We note that this is possible, for example, if $\overline{\Omega} \ni x \mapsto \sigma(x, \cdot) \in L^\infty(V)$ is piecewise continuous.

Lemma 3.13. *Under the assumptions of Theorem 3.10,*

$$\|K^*R(\nu + i\gamma, T)K\| \to 0 \text{ and } \|KR(\nu + i\gamma, T)K^*\| \to 0 \text{ as } |\gamma| \to \infty$$

for all $\nu > 0$.

Proof. First, we note the following continuous dependence of $R(\nu + i\gamma, T)$ on the collision frequency. Let T, $\widetilde{T}$ be associated with σ, $\widetilde{\sigma} \in L^\infty(\Omega \times V)$, respectively. Then

$$\|R(\nu + i\gamma, T) - R(\nu + i\gamma, \widetilde{T})\| \leq C_\nu \|\sigma - \widetilde{\sigma}\|_\infty$$

for some positive constant depending only on ν. Thus we may assume that σ is of the degenerate form, i.e. $\sigma(x,v) = \sum_{i\in I} \sigma_1^i(x)\sigma_2^i(v)$ with I finite. On the other hand, the operators $K^*R(\nu+i\gamma,T)K$ and $KR(\nu+i\gamma,T)K^*$, depend linearly and continuously, in the operator norm topology, on the collision operator K uniformly for $\gamma \in \mathbb{R}$, then, according to Lemma 3.2, it suffices to prove that

$$\|K_1R(\nu+i\gamma,T)K_2\| \to 0 \text{ as } |\gamma| \to \infty,$$

where K_i, $i=1,2$, has the form

$$K_i : L^2(\Omega\times V) \ni \varphi \mapsto \int_V \alpha_i(x) f_i(v) g_i(v')\varphi(x,v')dv' \in L^2(\Omega\times V)$$

with $\alpha_i \in L^\infty(\Omega)$ and f_i, $g_i \in L^2(V)$. Moreover, by density one may assume that f_i and g_i are continuous functions with supports in $V \cap \{v; \delta \le |v| \le 1/\delta\}$ for some fixed $0<\delta<1$. In this case, one easily sees that $K_1R(\nu+i\gamma,T)K_2$ is decomposable as

$$K_1R(\nu+i\gamma,T)K_2 = OQ(\gamma)P$$

with

$$P : L^2(\Omega\times V) \ni \varphi \mapsto \alpha_2(x)\int_V g_2(v)\varphi(x,v)dv \in L^2(\Omega),$$

$$O : L^2(\Omega) \ni \psi \mapsto \alpha_1(x)f_1(v)\psi(x) \in L^2(\Omega\times V)$$

and

$$Q(\gamma) : L^2(\Omega) \ni \varphi \mapsto \int_V \int_0^{\tau(x,v)} e^{-t(\nu+i\gamma)-\int_0^t \sigma(x-sv,v)ds} h(v)\varphi(x-tv)dtdv \in L^2(\Omega),$$

where $h = g_1f_2$. By the change of variable $y = x - tv$ we get

$$Q(\gamma)\varphi(x) = \int_{\mathbb{R}^N}\int_0^\infty e^{-t(\nu+i\gamma)-\int_0^t \sigma(x-s\frac{x-y}{t},\frac{x-y}{t})ds} h\big(\frac{x-y}{t}\big)\chi(x,x-y)\varphi(y)\frac{dt}{t^N}dy,$$

where all functions are extended by zero outside of their domains and

$$\chi(x,v) := \begin{cases} 1 & \text{if } 1 < \tau(x,v), \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$N_\gamma(x,z) = \int_0^\infty e^{-t(\nu+i\gamma)-\int_0^t \sigma(x+z-s\frac{x}{t},\frac{x}{t})ds} h\big(\frac{x}{t}\big)\frac{dt}{t^N}.$$

Then one sees that

$$\begin{aligned} |Q(\gamma)\varphi(x)| &\le \int_{\mathbb{R}^N} |N_\gamma(x-y,y)||\varphi(y)|dy \\ &\le \int_{\mathbb{R}^N} \sup_{z\in\mathbb{R}^N} |N_\gamma(x-y,z)||\varphi(y)|dy \\ &\le \big(\sup_{z\in\mathbb{R}^N}|N_\gamma(\cdot,z)|\big) * |\varphi(\cdot)|(x), \end{aligned}$$

so by [4, Theorem IV.15]

$$\|Q(\gamma)\| \leq \| \sup_{z\in\mathbb{R}^N} |N_\gamma(\cdot,z)|\|_{L^1(\mathbb{R}^N)}.$$

Thus it remains to show

$$\| \sup_{z\in\mathbb{R}^N} |N_\gamma(\cdot,z)|\|_{L^1(\mathbb{R}^N)} \to 0 \text{ as } |\gamma| \to \infty.$$

First, observe that

$$N_\gamma(x,z) = \int\limits_{-\infty}^{\infty} \frac{\chi_{[0,\infty[}(t)}{t^N} e^{-i\gamma t} e^{-t\nu - t\left(\sum_{i\in I}\sigma_2^i(\frac{x}{t})\int\limits_0^1 \sigma_1^i(x+z-sx)ds\right)} h(\frac{x}{t})dt$$

and that for every $x \in \mathbb{R}^N$ the set

$$\{S_{x,z} \in L^1(\mathbb{R});\ z \in \mathbb{R}^N\},$$

where

$$S_{x,z} : t \in \mathbb{R} \mapsto \frac{\chi_{[0,\infty[}(t)}{t^N} e^{-t\nu - t\sum_{i\in I}\sigma_2^i(\frac{x}{t})\int\limits_0^1 \sigma_1^i(x+z-sx)ds} h(\frac{x}{t}),$$

is relatively compact in $L^1(\mathbb{R})$. Indeed, let $(z_n)_n$ be a sequence in $\mathbb{R}^N$. Pick a subsequence $(z_{n_k})_k$ such that

$$\int\limits_0^1 \sigma_1^i(x+z_{n_k}-sx)ds \to c_x^i \text{ as } k\to\infty \text{ for all } i \in I.$$

By the dominated convergence theorem the sequence $(S_{x,z_{n_k}})_k$ converges, in $L^1(\mathbb{R})$, to the function

$$t \in \mathbb{R} \mapsto \frac{\chi_{[0,\infty[}(t)}{t^N} e^{-t\nu - t\sum_{i\in I}\sigma_2^i(\frac{x}{t})c_x^i} h(\frac{x}{t}).$$

So according to Riemman-Lebesgue's lemma we get

$$\int\limits_0^\infty e^{-i\gamma t} e^{-t\nu - t\int\limits_0^1 \sigma(x+z-sx,\frac{x}{t})ds} h(\frac{x}{t})\frac{dt}{t^N} \to 0 \text{ as } |\gamma|\to\infty$$

uniformly for $z \in \mathbb{R}^N$, i.e.

$$\sup_{z\in\mathbb{R}^N} |N_\gamma(x,z)| \to 0 \text{ as } |\gamma| \to \infty$$

for all x. By the dominated convergence theorem again we conclude that

$$\| \sup_{z\in\mathbb{R}^N} |N_\gamma(\cdot,z)|\|_{L^1(\mathbb{R}^N)} \to 0 \text{ as } |\gamma| \to \infty,$$

which ends the proof. ∎

Remark 3.14. The results of this section can be extended, by density arguments (Lemma 3.2) and interpolation, to L^p spaces $(1 < p < \infty)$.

3.4 Further Extensions

In neutron transport equations it may happen that the collision operator K is a sum of several operators. This occurs in nuclear reactor theory for solid moderators, where the collision operator is a sum of (an incoherent part) K_i and (a coherent part) K_c where K_i is compact in $L^2(V)$ while K_c is not [5]. It is also the case in [12, 23] where K consists of three terms: the first, K_c, is that given by (1.10), the second term, K_d, is a singular nilpotent operator, and the third term, K_0, describes the elastic scattering which is "not compact in velocity". On the other hand if, for instance, $K \in \mathcal{L}(L^2(V))$ is not power compact then the terms $U_n(t)$ are never compact (see [17, Chapter 4]). These facts motivate M. Mokhtar-Kharroubi to state the following problem ([17, Problem 3, p. 93]): Let $K = K_1 + K_2$ be a collision operator such that K_2 is regular. Find (an estimate of) the essential type of the semigroup $(V(t))_{t\geq 0}$ generated by $T + K$.
Before answering this problem, let us first remark that, for ν large,

$$\begin{aligned} & K_2R(\nu + i\gamma, T + K_1) \\ = \; & \big(K_2R(\nu + i\gamma, T)(\nu + i\gamma - T - K_1) + K_2R(\nu + i\gamma, T)K_1\big)R(\nu + i\gamma, T + K_1) \\ = \; & K_2R(\nu + i\gamma, T) + K_2R(\nu + i\gamma, T)K_1R(\nu + i\gamma, T + K_1). \end{aligned} \tag{3.5}$$

Similarly,

$$R(\nu + i\gamma, T + K_1)K_2 = R(\nu + i\gamma, T)K_2 + R(\nu + i\gamma, T + K_1)K_1R(\nu + i\gamma, T)K_2. \tag{3.6}$$

Let us denote by $(W(t))_{t\geq 0}$ the C_0-semigroup generated by $T + K_1$. We have:

Theorem 3.15. *Let $p = 2$ and let $K = K_1 + K_2$ be a collision operator such that K_2 is regular.*
(a) *Let μ be such that the affine hyperplanes have zero μ-measure.*
• *If Ω is convex and bounded and if the collision frequency is homogenous, then $\sigma_{ess}(V(t)) = \sigma_{ess}(W(t))$ for all $t \geq 0$.*
• *If Ω is convex and if the collision frequency is homogenous, then $\sigma_{crit}(V(t)) = \sigma_{crit}(W(t))$ for all $t \geq 0$.*
(b) *Let μ be the Lebesgue measure, and let the collision frequency be as in Theorem 3.10. Then $\sigma_{crit}(V(t)) = \sigma_{crit}(W(t))$ for all $t \geq 0$.*

Proof. We start with the second part of **(a)**. Using Lemma 3.3, (3.5) and (3.6), we get

$$\|K_2R(\nu + i\gamma, T + K_1)\| + \|R(\nu + i\gamma, T + K_1)K_2\| \to 0 \text{ as } |\gamma| \to \infty,$$

which proves, by invoking Theorem 2.2, the second part of **(a)**. Moreover, if Ω is bounded, then according to [17, Theorem 4.1, p. 57] $K_2R(\nu + i\gamma, T)$ is compact, and hence by (3.5), $K_2R(\nu + i\gamma, T + K_1)$ is compact too. Theorem 2.3 ends then the proof of the first part of **(a)**. Since T is dissipative, by Lemma 2.8 and Lemma 3.13 we have

$$\|K_2R(\nu + i\gamma, T)\| + \|R(\nu + i\gamma, T)K_2\| \to 0 \text{ as } |\gamma| \to \infty \text{ for all } \nu > 0.$$

Then by (3.5) and (3.6)

$$\|K_2R(\nu + i\gamma, T + K_1)\| + \|R(\nu + i\gamma, T + K_1)K_2\| \to 0 \text{ as } |\gamma| \to \infty \text{ for all } \nu > 0.$$

Finally, Theorem 2.2 ends the proof of **(b)**. ∎

Remark 3.16. If Ω is convex and bounded, Theorem 3.15 provides, in particular, an estimate for the essential type of $(V(t))_{t\geq 0}$: $\omega_{ess}(V(\cdot)) = \omega_{ess}(W(\cdot))$. Actually, if K_1 is positive in the lattice sense, then by the L. Weis's result [32] on the identity of the type of positive semigroups on L^p spaces and the spectral bound of its generators we have $\omega_{ess}(V(\cdot)) \leq s(T + K_1)$ (the spectral bound of $T + K_1$). The latter can be an equality (see [23]).

Remark 3.17. We point out that Theorem 3.15 holds regardless of the nature of the operator K_1. This shows how much the resolvent approach can appear powerful for dealing with the stability of the essential and critical spectra, especially when the unperturbed semigroup is not explicit.

Remark 3.18. By interpolation arguments, Theorem 3.15 remains true in L^p-space $(1 < p < \infty)$ if K_2 is regular and if $K_1 \in \bigcap_{q\geq 1} \mathcal{L}(L^q(\Omega \times V))$ or it can be approximated by such operators. See [23] for collision operators satisfying this property.

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